VINCIA Authors' Compendium

P. Skands, N. Fischer, A. Lifson

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Contents

A Final–Final Evolution Equations

A.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $IK \to ijk$, respectively. For massless partons, note the relations (and notation):

$$E_{\rm cm}^2 \equiv m_{\rm Ant}^2 \equiv s_{IK} \equiv s_{ijk} = s_{ij} + s_{jk} + s_{ij} \tag{A.1}$$

and

$$x_j = \frac{2E_j}{\sqrt{s_{IK}}} = 1 - \frac{s_{ik}}{s_{IK}} \; . \tag{A.2}$$

For massive partons, we generally use the notation m for invariant masses and s for dot products, hence e.g., $m_{IK}^2 = (p_I + p_K)^2 = m_I^2 + m_K^2 + 2p_I \cdot p_K \equiv m_I^2 + m_K^2 + s_{IK}$, so that $s_{IK} \equiv 2p_I \cdot p_K$. The relation for massive particles is thus:

$$m_{\rm Ant}^2 = m_{IK}^2 = m_{ijk}^2 = s_{IK} + m_I^2 + m_K^2 = s_{ij} + s_{jk} + s_{ik} + m_i^2 + m_j^2 + m_k^2 \; . \tag{A.3}$$

We define the scaled (dimensionless) invariants by

$$y_{ij} \equiv \frac{s_{ij}}{s_{IK}} \; = \; \frac{m_{ij}^2 - m_i^2 - m_j^2}{m_{IK}^2 - m_I^2 - m_K^2} . \tag{A.4}$$

We also use the notation $\mu_i^2 = m_i^2 / s_{IK}$ for scaled masses.

For massless partons as well as for the emission of a massless parton (like a photon or gluon) from arbitrary emitters I and K (with $m_i = m_I$ and $m_k = m_K$), the momentum conservation relation yields:

$$y_{ij} + y_{jk} + y_{jk} = 1 \; . \tag{A.5}$$

For the breakup of a massless gluon (or photon), I, to a massive quark-antiquark pair, i and j (with $m_i = m_j = m_Q$ and arbitrary recoiler mass $m_k = m_K$), the relation is:

$$y_{ij} + y_{jk} + y_{ik} + 2\mu_Q^2 = 1 \; . \tag{A.6}$$

For the emission of a massive particle, j, which does not change the flavour of the emitting parent (like emission of a Z or Higgs boson), we have for general m_j, $m_i = m_I$, and $m_k = m_K$:

$$y_{ij} + y_{jk} + y_{ik} + \mu_j^2 = 1 \tag{A.7}$$

Finally, for the emission of a massive particle, j, which does change the flavour of the emitting parent, we have for general m_j, $m_i \neq m_I$, and $m_k = m_K$:

$$y_{ij} + y_{jk} + y_{ik} + \mu_j^2 + \mu_i^2 - \mu_I^2 = 1 \tag{A.8}$$

A.2 Phase-Space Factorisation and Källén Factor

The FF antenna phase space, for general masses, is

$$\frac{\mathrm{d}\Phi_3^{ijk}}{\mathrm{d}\Phi_2^{IK}} = \frac{1}{16\pi^2}\,\frac{1}{\sqrt{\lambda(m_{IK}^2, m_I^2, m_K^2)}}\,\mathrm{d}s_{ij}\mathrm{d}s_{jk}\frac{\mathrm{d}\phi}{2\pi} \tag{A.9}$$

$$= \frac{1}{16\pi^2}\,\frac{s_{IK}^2}{\sqrt{\lambda(m_{IK}^2, m_I^2, m_K^2)}}\,\mathrm{d}y_{ij}\mathrm{d}y_{jk}\frac{\mathrm{d}\phi}{2\pi} \tag{A.10}$$

$$\overset{m_I\to 0\,||\,m_K\to 0}{=} \frac{1}{16\pi^2}\,s_{IK}\,\mathrm{d}y_{ij}\mathrm{d}y_{jk}\frac{\mathrm{d}\phi}{2\pi}\,, \tag{A.11}$$

with the Källén function $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + bc + ac)$ expressing the volume of the 2-particle phase space of the parent antenna.

Note that $\sqrt{\lambda} \to s_{IK}$ if either one or both of $m_{I,K} \to 0$. This is one reason we choose to normalise our dimensionful scales by s_{IK} rather than by m_{IK}^2. (Also, we regard it as convenient that in the limit that both of the two partons are heavy and at rest, the true size of phase space vanishes, but s_{IK} still goes to a finite number. s_{IK} thereby represents a nice intermediate choice between m_{IK}^2 and $\sqrt{\lambda}$.)

To account for the size of the 2-particle phase space for general FF antennae, we define the following "Källén factor" for each such antenna,

$$f_{\text{Källén}} \equiv \frac{s_{IK}}{\sqrt{\lambda(m_{IK}^2, m_I^2, m_K^2)}} \tag{A.12}$$

which is equal to unity except in the special case that both parents have non-zero masses (in which case it is bounded from below by unity and is used as an overall constant factor multiplying the trial antenna probability density).

The phase-space factorisation is then:

$$\frac{\mathrm{d}\Phi_3^{ijk}}{\mathrm{d}\Phi_2^{IK}} = \frac{1}{16\pi^2}\,f_{\text{Källén}}\,\frac{\mathrm{d}s_{ij}\mathrm{d}s_{jk}}{s_{IK}}\frac{\mathrm{d}\phi}{2\pi} \tag{A.13}$$

$$= \frac{1}{16\pi^2}\,s_{IK}\,f_{\text{Källén}}\,\mathrm{d}y_{ij}\mathrm{d}y_{jk}\frac{\mathrm{d}\phi}{2\pi}\,, \tag{A.14}$$

For a generic FF antenna function written as $\mathcal{C}g^2\bar{a}/s_{IK}$, with $g^2 = 4\pi\alpha$ a generic coupling factor, $\mathcal{C}$ a generic charge factor[1], and $\bar{a}$ a function of the scaled (dimensionless) invariants and masses, we thus have the generic integrand

$$\mathcal{C}\,g^2\,\frac{\bar{a}}{s_{IK}}\,\frac{\mathrm{d}\Phi_3^{ijk}}{\mathrm{d}\Phi_2^{IK}} = f_{\text{Källén}}\,\frac{\mathcal{C}\,\alpha}{4\pi}\,\bar{a}(y_{ij}, y_{jk}, \mu_i^2, \mu_j^2, \mu_k^2)\,\mathrm{d}y_{ij}\mathrm{d}y_{jk}\frac{\mathrm{d}\phi}{2\pi}\,. \tag{A.15}$$

[1]Note that we use a normalisation in which $\mathcal{C} = 1$ for photon emission off leptons, $\mathcal{C} = 2/3$ for photon emission off up quarks, $\mathcal{C} = 2C_F$ for gluon emission off quarks, $\mathcal{C} = C_A$ for gluon emission off gluons, and $\mathcal{C} = 1$ gluon splittings to quark-antiquark pairs.

A.3 Gluon Emission

A.3.1 pT Ordering

We define $p_\perp$ as

$$p_\perp^2 \;=\; \frac{(m_{ij}^2 - m_I^2)(m_{jk}^2 - m_K^2)}{m_{IK}^2 - m_I^2 - m_K^2} \;=\; \frac{s_{ij}s_{jk}}{s_{IK}}\;, \tag{A.16}$$

where the first expression is our general definition of $p_\perp$ and the second is valid for gluon emissions. The corresponding dimensionless invariant is:

$$x_\perp = \frac{p_\perp^2}{s_{IK}} = y_{ij}y_{jk}\;, \tag{A.17}$$

The upper limit for this variable on the physical branching phase space is:

$$p_{\perp\max}^2 \;\leq\; \frac{s_{IK}}{4} \quad ; \quad x_{\perp\max} \;\leq\; \frac{1}{4}\;, \tag{A.18}$$

with the maximum value reached for the point $(y_{ij}, y_{jk}) = (\frac{1}{2}, \frac{1}{2})$.

We define the complementary phase-space variable as the rapidity of the emitted gluon

$$y = \frac{1}{2}\ln\left(\frac{y_{jk}}{y_{ij}}\right)\;, \tag{A.19}$$

with the inverse transformations:

$$y_{ij} = e^{-y}\sqrt{x_\perp} \quad , \quad y_{jk} = e^{y}\sqrt{x_\perp} \tag{A.20}$$

Contours of constant $x_\perp$ and y values are shown in fig. 1 on linear (left) and logarithmic (right) scales, with the radiation pattern of a $q\bar{q} \to qg\bar{q}$ antenna superimposed in shades from blue (low probability) to yellow (high probability).

Trial Function: We use the universal soft-eikonal term of the antenna functions to generate trial emissions. This overestimates the physical antenna functions, which can therefore be recovered by a simple veto procedure. (The only exception is when matrix-element corrections are used, in which case larger trial overestimates may be needed, up to the matrix-element-corrected orders.) The trial emission (eikonal) antenna function is

$$\bar{a}_E \;=\; \frac{2}{y_{ij}y_{jk}} \;=\; \frac{2}{x_\perp}\;. \tag{A.21}$$

Trial Integrals: The trial integral is, symbolically,

$$\mathcal{A}_E = f_{\text{Källén}}\frac{\mathcal{C}}{2\pi}\int_{Q_2^2}^{Q_1^2} \alpha\,\frac{\mathrm{d}y_{ij}\mathrm{d}y_{jk}}{y_{ij}y_{jk}}\;. \tag{A.22}$$

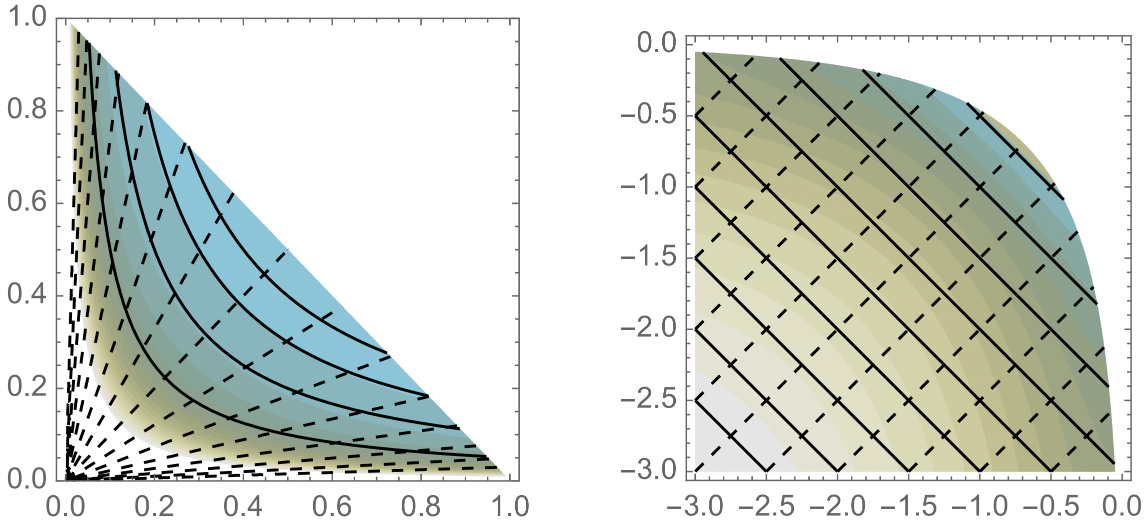

Figure 1: Left: equidistant contours of constant $x_\perp$ and y (with 0.05 between $x_\perp$ contours and 0.25 between y ones) functions of y_{ij} and y_{jk}. Right: equidistant contours of constant $\ln(x_\perp)$ and y (with 0.5 between $\ln(x_\perp)$ contours and 0.25 between y ones), as functions of $\ln y_{ij}$ and $\ln y_{jk}$. Note that the latter, rotated counterclockwise by 45 degrees, is also called the Lund plane.

The Jacobian for transforming to $(x_\perp, y)$ is unity:

$$\mathrm{d}y_{ij}\mathrm{d}y_{jk} = \mathrm{d}x_\perp \mathrm{d}y \, , \tag{A.23}$$

so that the trial integral becomes

$$\mathcal{A}_E = f_{\text{Källén}} \frac{\mathcal{C}}{2\pi} \int_{x_{\perp 2}}^{x_{\perp 1}} \alpha \frac{\mathrm{d}x_\perp}{x_\perp} \int_{y_-(x_\perp)}^{y_+(x_\perp)} \mathrm{d}y \tag{A.24}$$

The exact rapidity integral is given by the physical rapidity range

$$y_\pm^{\mathrm{phys}} = \pm \frac{1}{2} \ln \left(\frac{1+\sqrt{1-4x_\perp}}{1-\sqrt{1-4x_\perp}} \right) \, , \tag{A.25}$$

but since this would yield too cumbersome expressions to work with in practice, we shall use simple overestimates of this range and recover the physical one by vetos.

For a constant trial α, we overestimates the rapidity range by the Lund triangle limits:

$$y_\pm^{\mathrm{Lund}} = \pm \frac{1}{2} \ln(s_{IK}/p_\perp^2) = \pm \frac{1}{2} \ln(1/x_\perp) \implies \Delta y^{\mathrm{Lund}}(x_\perp) = \ln(1/x_\perp) \, . \tag{A.26}$$

Inserting this overestimate, the trial integral becomes

$$\mathcal{A}_E = -f_{\text{Källén}} \frac{\mathcal{C}}{2\pi} \int_{x_{\perp 2}}^{x_{\perp 1}} \alpha \frac{\mathrm{d}x_\perp}{x_\perp} \ln(x_\perp) \tag{A.27}$$

$$= \alpha f_{\text{Källén}} \frac{\mathcal{C}}{4\pi} \left(\ln^2(x_{\perp 2}) - \ln^2(x_{\perp 1}) \right) \, . \tag{A.28}$$

Solving the equation $R = \exp(-\mathcal{A}_E)$ for x_2 yields:

$$\textbf{Constant trial } \alpha_s: \quad \ln^2 x_{\perp 2} = \ln^2 x_{\perp 1} - \frac{4\pi}{\alpha f_{\text{Källén}} \mathcal{C}} \ln R \,. \tag{A.29}$$

With an $x_\perp$ generated according to this expression, a trial rapidity uniformly distributed on the interval defined by the Lund triangle is generated, and vetoed if it falls outside the physical rapidity range for the given $x_\perp$ value.

For an explicitly first-order running α_s, we overestimate the physical rapidity range at $x_\perp$ by the larger range accessible at some fixed $x_{\perp\min}$ corresponding to the lowest $x_\perp$ scale accessible in the current "evolution window",

$$y_\pm(x_{\perp\min}) = \pm\frac{1}{2}\ln\left(\frac{1+\sqrt{1-4x_{\perp\min}}}{1-\sqrt{1-4x_{\perp\min}}}\right) \quad\Longrightarrow\quad \Delta y(x_{\perp\min}) = \ln\left(\frac{1+\sqrt{1-4x_{\perp\min}}}{1-\sqrt{1-4x_{\perp\min}}}\right) \,. \tag{A.30}$$

Defining

$$\begin{aligned} 1/\alpha_s &= b_0 \ln\left(k_R p_\perp^2/\Lambda^2\right) && \text{(A.31)}\\ &= b_0 \ln\left(k_R x_\perp s_{IK}/\Lambda^2\right) && \text{(A.32)}\\ &= b_0 \ln\left(x_\perp/x_\Lambda\right) \,, && \text{(A.33)} \end{aligned}$$

with $x_\Lambda = \Lambda^2/(k_R s_{IK})$, the trial integral becomes:

$$\begin{aligned} \mathcal{A}_E &= f_{\text{Källén}} \frac{\mathcal{C}}{2\pi b_0} \Delta y(x_{\perp\min}) \int_{x_{\perp 2}}^{x_{\perp 1}} \frac{\mathrm{d}x_\perp}{x_\perp} \frac{1}{\ln(x_\perp/x_\Lambda)} && \text{(A.34)}\\ &= f_{\text{Källén}} \frac{\mathcal{C}}{2\pi b_0} \Delta y(x_{\perp\min}) \int_{\ln x_{\perp 2}/x_\Lambda}^{\ln x_{\perp 1}/x_\Lambda} \frac{\mathrm{d}\ln(x_\perp/x_\Lambda)}{\ln(x_\perp/x_{\perp 0})} && \text{(A.35)}\\ &= f_{\text{Källén}} \frac{\mathcal{C}}{2\pi b_0} \Delta y(x_{\perp\min}) \int_{\ln\ln x_{\perp 2}/x_\Lambda}^{\ln\ln x_{\perp 1}/x_\Lambda} \mathrm{d}\ln\ln(x_\perp/x_\Lambda) && \text{(A.36)}\\ &= f_{\text{Källén}} \frac{\mathcal{C}}{2\pi b_0} \Delta y(x_{\perp\min}) \left[\ln\ln\left(\frac{x_{\perp 1}}{x_\Lambda}\right) - \ln\ln\left(\frac{x_{\perp 2}}{x_\Lambda}\right)\right] \,. && \text{(A.37)} \end{aligned}$$

Trial Generation: Solving $R = \exp(-\mathcal{A}_E)$ for $p_{\perp 2}^2$ (with R a uniformly distributed random number on the interval $[0,1]$), we obtain:

$$\textbf{Running trial } \alpha_s: \quad p_{\perp 2}^2 = \frac{\Lambda^2}{k_R}\left(\frac{p_{\perp 1}^2}{(\Lambda^2/k_R)}\right)^{R^{(b_0/I_E)}} \,, \tag{A.38}$$

with

$$I_E = \frac{f_{\text{Källén}} \mathcal{C} \Delta y(x_{\perp\min})}{2\pi} \tag{A.39}$$

With an $x_\perp$ generated according to this expression, a trial rapidity uniformly distributed on the interval defined by $x_{\perp\min}$ is generated, and vetoed if it falls outside the physical rapidity range for the given $x_\perp$ value.

A.3.2 pTmin Ordering

In conventional pT ordering, as above, the measured quantity is always defined as the pT of the emitted gluon with respect to the parent antenna.

However, if a recoiling parent becomes soft (eg via a collinear branching where the "emitted" gluon takes most of the energy), then there is both an ambiguity about what constitutes "the parent antenna", and, if the parent that becomes soft is a gluon, a neighbouring antenna may "accidentally" acquire a small pT resolution scale after the branching, simply due to the momentum-conservation / recoil effect.

An alternative is to define the evolution scale as the smallest resolution scale of any of the post-branching partons with respect to each other, a choice we refer to as $p_{\perp\mathrm{min}}$. The effects of these choices have not so far been well studied. Note that this choice differs from the "sector antenna" approach in that it is still only the local $2 \to 3$ branching variables that are used to define the scale, without reference to the neighbouring partons' momenta. The definition of $p^2_{\perp\mathrm{min}}$ is:

$$p^2_{\perp\mathrm{min}} = s_{IK}\mathrm{min}(y_{ij}y_{jk}, y_{jk}y_{ik}, y_{ik}y_{ij}) \ . \tag{A.40}$$

The upper limit for this variable on the physical branching phase space is:

$$p^2_{\perp\mathrm{min}} \ \leq \ \frac{s_{IK}}{9} \quad ; \quad x_{\perp\mathrm{max}} \ \leq \ \frac{1}{9} \ , \tag{A.41}$$

with the maximum value reached for the point $(y_{ij}, y_{jk}) = (\frac{1}{3}, \frac{1}{3})$; the Mercedes point.

Equidistant contours of $p_{\perp\mathrm{min}}$ and of the complementary phase-space variable ζ (see below) are plotted in fig. 2, on linear (left) and logarithmic (right) scales, with the radiation pattern of a $q\bar{q} \to qg\bar{q}$ antenna superimposed in shades from blue (low probability) to yellow (high probability). As can clearly be seen from the plots, it is the behaviour in the two hard-collinear regions that is modified (as well as the behaviour in the nonsingular hard region in which the invariant between the two parent partons can become accidentally small), while the behaviour for soft-gluon emission is not modified. The point of symmetry is the Mercedes point, which constitutes the "most resolved" branching according to this evolution variable. The main difference with respect to evolution in the conventional pT variable should be to "push" hard-collinear branchings to occur at lower evolution scales. Such branchings will therefore receive more significant Sudakov suppression, while branchings in the soft and wide-angle region will receive slightly less suppression.

We divide the trial generation into three regions,

$$\text{Region J} \ : \ y_{ij}y_{jk} \quad < \quad y_{ik}\,\mathrm{min}(y_{ij}, y_{jk}) \tag{A.42}$$

$$\text{Region I} \ : \ y_{ij}y_{ik} \quad < \quad y_{jk}\,\mathrm{min}(y_{ij}, y_{ik}) \tag{A.43}$$

$$\text{Region K} \ : \ y_{ik}y_{jk} \quad < \quad y_{ij}\,\mathrm{min}(y_{ik}, y_{jk}) \tag{A.44}$$

corresponding to each of the three $p_\perp$ definitions.

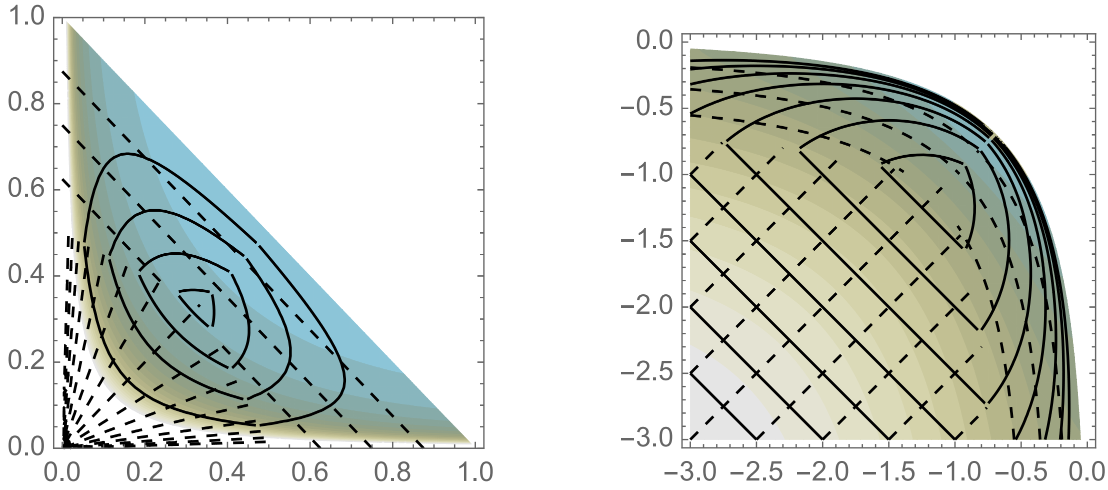

Figure 2: Left: equidistant contours of constant $x_{\perp\text{min}}$ and corresponding ζ (with 0.025 between $x_{\perp\text{min}}$ contours and 0.25 between ζ ones) functions of y_{ij} and y_{jk}. Right: equidistant contours of constant $\ln(x_{\perp\text{min}})$ and y (with 0.5 between $\ln(x_{\perp\text{min}})$ contours and 0.25 between ζ ones), as functions of $\ln y_{ij}$ and $\ln y_{jk}$.

The Soft-Gluon Region, J: In region J, we can use the same trial generation machinery as for the conventional pT ordering, with the $p_{\perp j} < \min(p_{\perp i}, p_{\perp k})$ criterion imposed by veto. To make things slightly more efficient, we note that the trial generation in region J can be restricted to the smaller rapidity range

$$\text{Region J} \ : \ y_{\pm}^{\text{trial}} \ = \ \pm\frac{1}{2}\ln\left(\frac{1}{4x_{\perp j}}\right) \ , \tag{A.45}$$

$$\Delta y^{\text{trial}} \ = \ \ln\left(\frac{1}{4x_{\perp j}}\right) \ = \ \ln\left(\frac{1}{x_{\perp j}}\right) - \ln(4) \ . \tag{A.46}$$

This can be used with the formalism for either a constant or first-order running trial α_s.

The I- and K-Collinear Regions: In the two other regions, we use a collinear trial-function overestimate. For regions I and K (in which partons i and k, respectively, are considered the softest partons), it is:

$$\bar{a}_{Ei} \ = \ \frac{4}{y_{ij}} \tag{A.47}$$

$$\bar{a}_{Ek} \ = \ \frac{4}{y_{jk}} \tag{A.48}$$

With these functions, it is convenient to choose $\zeta = 2y_{ik}$ as the complementary phase-space variable, whose maximal range is $[0, 1]$ in each of the I- and K-collinear regions. A better lower bound can be obtained by noting that the lowest ζ value for a given $x_\perp$ is obtained for the point $y_{ij} = y_{jk}$, yielding

$$\zeta_{\rm min}(x_\perp) \;=\; 1 - \sqrt{1-8x_\perp} \;\geq\; 4x_\perp\;. \tag{A.49}$$

The Jacobian for transforming from (y_{ij}, y_{jk}) to $(x_\perp, \zeta)$ in the I- and K-collinear regions is then $|J| = 1/(2y_{ik})$, so that the trial function in the transformed coordinates is again just:

$$\bar{a}_{Ei}|J| \;=\; \bar{a}_{Ek}|J| \;=\; \frac{2}{x_\perp}\;. \tag{A.50}$$

A trial ζ can then be generated uniformly on the interval $[\zeta_{\rm min}, 1]$ and rejected if it falls outside the physical phase space for the relevant region.

A.4 Gluon Splitting

A.4.1 pT Ordering

For gluon splittings, the definition of pT is somewhat more ambiguous than it is for gluon emissions. For instance, even if one uses the same functional form as is used for gluon emissions, which parton should be thought of as the "emitted" one? Is it the quark, or the antiquark, or some combination thereof? The first option we consider is to define pT ordering for gluon splittings to be as close to what is used for gluon emissions as possible. Thus, for a colour antenna in which the gluon acts as the anticolour source (e.g., gluon splitting in a qg antenna), we define the $p_\perp$ as that of the produced antiquark, with the produced quark taking the role of a recoiling parent. For a colour antenna in which the gluon acts as the colour source (e.g., gluon splitting in a $g\bar{q}$ antenna), we define the $p_\perp$ as that of the quark, with the antiquark acting as a recoiling parent.

Labeling the participating partons as $g_I X_K \to q_i \bar{q}_j X_k$, with X an arbitrary recoiler, we have $m_I = 0$, $m_i = m_j = m_q$, and $m_k = m_K$. Depending on whether it is the colour or anticolour, respectively, of the gluon which is active in the splitting, the corresponding $p_\perp^2$ scale is:

$$p_{\perp i}^2 \;=\; \frac{(m_{ij}^2 - m_I^2)(m_{ik}^2 - m_K^2)}{s_{IK}} \;=\; \frac{m_{ij}^2(m_{ik}^2 - m_k^2)}{m_{IK}^2 - m_k^2} \;=\; \frac{(s_{ij} + 2m_q^2)(s_{ik} + m_q^2)}{s_{IK}} \tag{A.51}$$

$$p_{\perp j}^2 \;=\; \frac{(m_{ij}^2 - m_I^2)(m_{jk}^2 - m_K^2)}{s_{IK}} \;=\; \frac{(m_{ij}^2(m_{jk}^2 - m_k^2)}{m_{IK}^2 - m_k^2} \;=\; \frac{(s_{ij} + 2m_q^2)(s_{jk} + m_q^2)}{s_{IK}}\;, \tag{A.52}$$

where the first equality just recalls our general definition of $p_\perp$ and the latter ones specialise to the case of gluon splitting.

As the complementary phase-space invariant ζ, we take the nonsingular invariant in the corresponding pT definition,

$$\zeta_i = y_{ik} + \mu_q^2 = \frac{s_{ik} + m_q^2}{s_{IK}} , \tag{A.53}$$

$$\zeta_j = y_{jk} + \mu_q^2 = \frac{s_{jk} + m_q^2}{s_{IK}} , \tag{A.54}$$

with the inverse relations:

$$s_{ij} = s_{IK}\frac{x_\perp}{\zeta} - 2m_q^2 \quad \Longrightarrow \quad m_{ij}^2 = s_{IK}\frac{x_\perp}{\zeta} , \tag{A.55}$$

$$s_{ik} = s_{IK}\zeta_i - m_q^2 , \tag{A.56}$$

$$s_{jk} = s_{IK}\zeta_j - m_q^2 . \tag{A.57}$$

Trial Function: The gluon splitting functions are all proportional to $1/(2m_{ij}^2)$, hence we use the trial overestimate:

$$\bar{a}_S = \frac{s_{IK}}{2m_{ij}^2} = \frac{1}{2(y_{ij} + 2\mu_q^2)} = \frac{\zeta}{2x_\perp} \tag{A.58}$$

Trial Integrals: The trial integral is, symbolically,

$$\mathcal{A}_S = f_{\text{Källén}}\frac{n_f T_R}{8\pi}\int_{Q_2^2}^{Q_1^2} \alpha \, \frac{\mathrm{d}y_{ij}\mathrm{d}y_{jk}}{(y_{ij} + 2\mu_q^2)} . \tag{A.59}$$

The Jacobian for transforming to $(x_\perp, \zeta)$ is $1/\zeta$:

$$\mathrm{d}y_{ij}\mathrm{d}y_{jk} = \mathrm{d}x_\perp\frac{\mathrm{d}\zeta}{\zeta} , \tag{A.60}$$

so that the trial integral becomes

$$\mathcal{A}_S = f_{\text{Källén}}\frac{n_f T_R}{8\pi}\int_{x_{\perp 2}^2}^{x_{\perp 1}^2} \alpha \, \frac{\mathrm{d}x_\perp}{x_\perp}\mathrm{d}\zeta . \tag{A.61}$$

The ζ limits can be overestimated by the constant range $[\mu_q^2, 1-3\mu_q^2]$, with the physical region (identified for the case of general masses by positivity of the Gram determinant) imposed by veto. Using this, we have:

$$\Delta\zeta^{\text{trial}} = 1 - 4\mu_q^2 . \tag{A.62}$$

For a constant trial α_s, the trial integral is thus:

$$\mathcal{A}_S = \alpha f_{\text{Källén}}\frac{n_f T_R}{8\pi}\Delta\zeta^{\text{trial}}\int_{x_{\perp 2}}^{x_{\perp 1}} \frac{\mathrm{d}x_\perp}{x_\perp} \tag{A.63}$$

$$= \alpha f_{\text{Källén}}\frac{n_f T_R}{8\pi}\Delta\zeta^{\text{trial}} \ln\left(\frac{x_{\perp 1}}{x_{\perp 2}}\right) . \tag{A.64}$$

Solving the equation $\mathcal{R} = \exp(-\mathcal{A}_S)$ yields:

$$\textbf{Constant trial } \alpha_s\textbf{ :} \quad x_{\perp 2} = x_{\perp 1} \mathcal{R}^{1/(\alpha I_S)} \tag{A.65}$$

with

$$I_S = \frac{f_{\text{Källén}} n_f T_R \Delta\zeta^{\text{trial}}}{8\pi} \ . \tag{A.66}$$

For a one-loop running trial α_s, the same integral as for gluon emission applies, modulo the colour factor and the overall factor 1/4:

$$\textbf{Running trial } \alpha_s\textbf{ :} \quad p_{\perp 2}^2 = \frac{\Lambda^2}{k_R} \left(\frac{p_{\perp 1}^2}{(\Lambda^2/k_R)} \right)^{R^{b_0/I_S}} \ . \tag{A.67}$$

With an $x_\perp$ generated according to either of these expressions, a trial ζ is generated as:

$$\zeta^{\text{trial}} = \mu_q^2 + \mathcal{R}_\zeta \Delta\zeta^{\text{trial}} \ , \tag{A.68}$$

to be vetoed if it falls outside the physical phase space (as given by positivity of the Gram determinant).

A.4.2 pTmin Ordering

As an alternative to the above, it is also possible to use the minimum of the quark or antiquark pT as the ordering variable.

A.4.3 mass Ordering

A third option is to use the invariant mass of the splitting gluon as the ordering variable.

A.5 Kinematics Construction

The trial value for the complementary phase-space invariant, here denoted ζ, is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\text{min}}, \zeta)}{I_\zeta(\zeta_{\text{min}}, \zeta_{\text{max}})} \ , \tag{A.69}$$

where the boundary values $(\zeta_{\text{min}}, \zeta_{\text{max}})$ must be the same as those that were used to evaluate the I_ζ integrals during the generation of the trial scale, i.e., they must correspond to the phase-space overestimate used for the trial generation.

The generated value of ζ can now be compared to the limits imposed by the physical phase space at the generated value of Q and a rejection imposed if the generated ζ value falls outside the phase space.

Inverting the expressions for $Q^2(s_{ij}, s_{jk})$ and $\zeta(s_{ij}, s_{jk})$, the set of branching invariants is found. The energies in the CM frame of the branching system can then be constructed from

$$E_i = \frac{m^2 - m_{jk}^2 + m_i^2}{2m} \tag{A.70}$$

$$E_j = \frac{m^2 - m_{ik}^2 + m_j^2}{2m} \tag{A.71}$$

$$E_k = \frac{m^2 - m_{ij}^2 + m_k^2}{2m} , \tag{A.72}$$

with $m^2 = m_{IK}^2$ the invariant mass squared of the antenna. The relative angles between the momenta are given by:

$$\cos\theta_{ij} = \frac{2E_iE_j + m_i^2 + m_j^2 - m_{ij}^2}{2|\vec{p}_i||\vec{p}_j|} \tag{A.73}$$

$$\cos\theta_{jk} = \frac{2E_jE_k + m_j^2 + m_k^2 - m_{jk}^2}{2|\vec{p}_j||\vec{p}_k|} . \tag{A.74}$$

$$\tag{A.75}$$

In terms of the dot-product $s_{ij} \equiv 2p_i \cdot p_j$ variables, the relations are:

$$m^2 = sij + sjk + sik + m2i + m2j + m2kE_i = \frac{s_{ij} + s_{ik} + 2m_i^2}{2m} \tag{A.76}$$

$$E_j = \frac{s_{jk} + s_{ij} + 2m_j^2}{2m} \tag{A.77}$$

$$E_k = \frac{s_{ik} + s_{jk} + 2m_k^2}{2m} , \tag{A.78}$$

$$\cos\theta_{ij} = \frac{2E_iE_j - s_{ij}}{2|\vec{p}_i||\vec{p}_j|} \tag{A.79}$$

$$\cos\theta_{jk} = \frac{2E_jE_k - s_{jk}}{2|\vec{p}_j||\vec{p}_k|} . \tag{A.80}$$

$$\tag{A.81}$$

The final orientation also depends on the choice of global recoil angle, ψ_{Ii}, which represents the angle between the pre- and post-branching partons I and i, around an axis perpendicular to the CM branching plane. Various specific forms can be chosen in the code, all of which must have the following collinear limits:

$$s_{jk} \to 0 \quad : \quad \psi_{Ii} \to 0 , \tag{A.82}$$

$$s_{ij} \to 0 \quad : \quad \psi_{Kk} = \psi_{Ii} + \theta_{ik} - \pi \to 0 . \tag{A.83}$$

A.6 Antenna Functions

For each antenna function, a full set of helicity-dependent antenna function contributions are implemented. For partons without helicity information, the unpolarised forms (summed over post-branching helicities and averaged over pre-branching ones) are used.

The functional forms given below omit colour and coupling factors. They are all normalised so that a factor $g_s^2 C_A = 4\pi\alpha_s C_A$ is appropriate in the leading-colour limit, with $C_A = 3$ replaced by $T_R = 1$ for gluon splittings. Corrections for subleading colour are discussed separately, in sec. A.7.

Some of the helicity-dependent antenna functions would not be positive definite over the full branching phase space if only the singular terms were included. In particular the emission of a gluon with opposite helicity to that of its parents can be negative for very hard emissions if only singular terms are included; nonsingular terms have then been added to render the full set of helicity-dependent antenna functions positive definite over all of phase space.

We here give only the forms for so-called "global" antenna functions, as sector antenna functions have not been fully implemented in the present VINCIA version.

The ij collinear limit of the functions can be studied by identifying $Q^2 = s_{ij} \to 0$ and

$$z_i = \frac{x_i}{x_i + x_j} = \frac{s_{IK} - s_{jk}}{s_{IK} + s_{ij}} \,, \tag{A.84}$$

thus

$$s_{ij} \;\to\; Q^2 \tag{A.85}$$

$$s_{jk} \;\to\; (1 - z_i)\, s_{IK} \tag{A.86}$$

$$s_{ik} \;\to\; z_i\, s_{IK} \tag{A.87}$$

and similarly for the ik collinear limit, with $i \leftrightarrow k$.

A.6.1 QQemitFF

The helicity-averaged antenna function is:

$$\begin{aligned} a(q_I q_K \to q_i g_j q_k) &= \frac{1}{s_{IK}} \left[\frac{2 y_{ik}}{y_{ij} y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} - \frac{2\mu_K^2}{y_{jk}^2} + \frac{y_{jk}}{y_{ij}} + \frac{y_{ij}}{y_{jk}} + 1 \right] \qquad \text{(A.88)} \\ &= \frac{1}{s_{IK}} \left[\frac{(1-y_{ij})^2 + (1-y_{jk})^2}{y_{ij} y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} - \frac{2\mu_K^2}{y_{jk}^2} + 1 \right] \\ &= A_3^0 + 1 \,. \qquad \text{(A.89)} \end{aligned}$$

Note: the A_3^0 function in GGG is derived from Z decay alone, while ours is the average of the Z^0 ones for $+-/-+$ ($J = 1$) parent configurations and and the H^0 ones for $++/--$ ($J = 0$)

ones. The difference between our antenna function and the GGG one is the $+1$ nonsingular term which is absent in the GGG A_3^0.

The collinear limits of this antenna function are:

$$a(y_{ij} \to 0, y_{jk} = 1 - x_i) \;=\; \frac{1}{m_{ij}^2 - m_I^2}\left[\frac{1+x_i^2}{1-x_i} - \frac{2m_I^2}{m_{ij}^2 - m_I^2}\right], \tag{A.90}$$

$$a(y_{ij} = 1 - x_k, y_{jk} \to 0) \;=\; \frac{1}{m_{jk}^2 - m_K^2}\left[\frac{1+x_k^2}{1-x_k} - \frac{2m_K^2}{m_{jk}^2 - m_K^2}\right] \tag{A.91}$$

The individual helicity contributions are (chosen such that all antenna functions remain positive definite over all of phase space):

$$a(++ \to +++) \;=\; \frac{1}{s_{IK}}\left[\frac{1}{y_{ij}y_{jk}} - \frac{\mu_i^2}{y_{ij}^2(1-y_{jk})} - \frac{\mu_k^2}{y_{jk}^2(1-y_{ij})}\right] \tag{A.92}$$

$$a(++ \to +-+) \;=\; \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^2 + (1-y_{jk})^2 - 1}{y_{ij}y_{jk}} + 2 - \frac{\mu_i^2(1-y_{jk})}{y_{ij}^2} - \frac{\mu_k^2(1-y_{ij})}{y_{jk}^2}\right] \tag{A.93}$$

$$a(++ \to -++) \;=\; \frac{1}{s_{IK}}\left[\frac{\mu_i^2 y_{jk}^2}{y_{ij}^2}\frac{1}{1-y_{jk}}\right] \tag{A.94}$$

$$a(++ \to ++-) \;=\; \frac{1}{s_{IK}}\left[\frac{\mu_k^2 y_{ij}^2}{y_{jk}^2}\frac{1}{1-y_{ij}}\right] \tag{A.95}$$

$$a(+- \to ++-) \;=\; \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^2}{y_{ij}y_{jk}} - \frac{\mu_i^2}{y_{ij}^2(1-y_{jk})} - \frac{\mu_k^2(1-y_{ij})}{y_{jk}^2}\right] \tag{A.96}$$

$$a(+- \to +--) \;=\; \frac{1}{s_{IK}}\left[\frac{(1-y_{jk})^2}{y_{ij}y_{jk}} - \frac{\mu_i^2(1-y_{jk})}{y_{ij}^2} - \frac{\mu_k^2}{y_{jk}^2(1-y_{ij})}\right] \tag{A.97}$$

$$a(+- \to -+-) \;=\; \frac{1}{s_{IK}}\left[\frac{\mu_i^2 y_{jk}^2}{y_{ij}^2(1-y_{jk})}\right] \tag{A.98}$$

$$a(+- \to +-+) \;=\; \frac{1}{s_{IK}}\left[\frac{\mu_k^2 y_{ij}^2}{y_{jk}^2(1-y_{ij})}\right]. \tag{A.99}$$

Note that the sum of the ++ antenna functions has the same singularities as the sum of the +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note that, for a scalar decay, the helicity-flip one ($++ \to +-+$) has to go to zero on the hard boundary ($x_j = 1-y_{ik} = y_{ij}+y_{jk} = 1$). In principle, one could (?) envision lifting this constraint (e.g., by adding a finite term that vanishes on the collinear boundaries, like $y_{ij}y_{jk}$) to account for t-channel processes which could produce a $++$ state with a higher total angular momentum, e.g., $J = 2$. Matrix-element corrections for scalar decay would then bring the matched result back to zero (which corresponds to the $+2$ finite term in the $+-+$ antenna), whiler MECs for t-channel processes could be non-zero. In the corresponding IF antenna, the finite term is instead chosen to be $+3 - y_{aj}^2 - y_{jk}^2$ which remains zero at the hard-collinear points but is nonzero along the diagonal.

A.6.2 QGemitFF

The helicity average (for unpolarised partons), for our default choice of gluon-collinear partitioning parameter $\alpha = 0$ (see below) is:

$$a(q_I g_K \to q_i g_j g_k)|_{\alpha=0} = \frac{1}{s_{IK}}\left[\frac{2y_{ik}}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} + \frac{y_{jk}}{y_{ij}} + \frac{y_{ij}(1-y_{ij})}{y_{jk}} + y_{ij} + \frac{y_{jk}}{2}\right]$$

$$= \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^3 + (1-y_{jk})^2}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} + \frac{y_{ik}-y_{ij}}{y_{jk}} + 1 + y_{ij} + \frac{y_{jk}}{2}\right] \tag{A.100}$$

$$\overset{\mu_I=0}{=} d_3^0 - \frac{5}{2} + 2y_{ij} + y_{jk} \,. \tag{A.101}$$

The collinear limits of this antenna function are:

$$a(y_{ij} \to 0, y_{jk} = 1 - x_i, \alpha = 0) = \frac{1}{m_{ij}^2 - m_I^2}\left[\frac{1+x_i^2}{1-x_i} - \frac{2m_I^2}{m_{ij}^2 - m_I^2}\right] \tag{A.102}$$

$$a(y_{ij} = 1 - x_k, y_{jk} \to 0, \alpha = 0) = \frac{1}{m_{jk}^2}\frac{2x_k + x_k(1-x_k)^2}{(1-x_k)}\,, \tag{A.103}$$

$$a(1 - x_k, y_{jk} \to 0) + a(x_k, y_{jk} \to 0, \alpha = 0) = \frac{2}{m_{jk}^2}\frac{(1 - x_k(1-x_k))^2}{x_k(1-x_k)} \tag{A.104}$$

Note 1: the D_3^0 function in GGG is derived from neutralino decay and contains a sum over both of the permutations of the gluons. Our function corresponds to the sub-antenna function d_3^0 from which it only differs by nonsingular terms.

Note 2: the singularity structure of the $qg \to qgg$ radiation function used in ARIADNE differs from ours by a term proportional to $(y_{ik} - y_{ij})$, which vanishes when summing over two neighbouring antennae (it is antisymmetric under interchange of the two gluons, j and k). Our parametrisation is chosen to agree with the GGG one which also has the property of minimising the sub-antenna contribution in the hard-collinear jk limit $y_{ij} \to 1$, corresponding to $x_k \to 1$.

(Those configurations are then maximally populated by the soft limit of the neighbouring antenna, which minimises the problem of recoils producing disparate pT scales.) For completeness, the helicity average obtained for $\alpha = 1$ (corresponding to the choice made in ARIADNE) is:

$$a(q_I g_K \to q_i g_j g_k)\big|_{\alpha=1} = \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^3+(1-y_{jk})^2}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} + 2 - y_{ij} - \frac{y_{jk}}{2}\right] \quad \text{(A.105)}$$

For general collinear-partitioning parameter α, the individual helicity contributions are:

$$a(++\to+++) = \frac{1}{s_{IK}}\left[\frac{1}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{jk})\left(\frac{1-2y_{ij}-y_{jk}}{y_{jk}}\right) - \frac{\mu_i^2}{y_{ij}^2(1-y_{jk})}\right], \quad \text{(A.106)}$$

$$a(++\to+-+) = \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})y_{ik}^2}{y_{ij}y_{jk}} - \frac{\mu_i^2(1-y_{jk})}{y_{ij}^2}\right], \quad \text{(A.107)}$$

$$a(++\to-++) = \frac{1}{s_{IK}}\left[\frac{\mu_i^2 y_{jk}^2}{y_{ij}^2(1-y_{jk})}\right], \quad \text{(A.108)}$$

$$a(+-\to++-) = \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^3}{y_{ij}y_{jk}} - \frac{\mu_i^2}{y_{ij}^2(1-y_{jk})}\right], \quad \text{(A.109)}$$

$$a(+-\to+--) = \frac{1}{s_{IK}}\left[\frac{(1-y_{jk})^2}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{jk})\left(\frac{1-2y_{ij}-y_{jk}}{y_{jk}}\right) - \frac{\mu_i^2(1-y_{jk})}{y_{ij}^2}\right], \quad \text{(A.110)}$$

$$a(+-\to-+-) = \frac{1}{s_{IK}}\left[\frac{\mu_i^2 y_{jk}^2}{y_{ij}^2(1-y_{jk})}\right]. \quad \text{(A.111)}$$

where again we remind that the GGG choice is obtained for $\alpha = 0$ while the ARIADNE one corresponds to $\alpha = 1$.

Note that the sum of the ++ antenna functions has the same singularities as the sum of the +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

A.6.3 GGemitFF

The helicity-averaged antenna function (for our default choice of gluon-collinear partitioning parameter $\alpha = 0$) is:

$$a(g_I g_K \to g_i g_j g_k) = \frac{1}{s_{IK}}\left[\frac{2y_{ik}}{y_{ij}y_{jk}} + \frac{y_{jk}(1-y_{jk})}{y_{ij}} + \frac{y_{ij}(1-y_{ij})}{y_{jk}} + \frac{1}{2}y_{ij} + \frac{1}{2}y_{jk}\right] \quad \text{(A.112)}$$

$$= \frac{1}{s_{IK}}\left[\frac{(1-y_{ij})^3+(1-y_{jk})^3}{y_{ij}y_{jk}} + \frac{y_{ik}-y_{ij}}{y_{jk}} + \frac{y_{ik}-y_{jk}}{y_{ij}} + 2 + \frac{1}{2}y_{ij} + \frac{1}{2}y_{jk}\right] \quad \text{(A.113)}$$

$$= f_3^0 - \frac{2}{3} - 2y_{ik} \quad \text{(A.114)}$$

$$= f_3^{0'} - 1 - y_{ik} \quad \text{(A.115)}$$

The collinear limits of this antenna function are:

$$a(y_{ij} = 1-x_k, y_{jk} \to 0) = \frac{1}{m_{jk}^2}\frac{2x_k + x_k(1-x_k)^2}{(1-x_k)}\ , \quad \text{(A.116)}$$

$$a(1-x_k, y_{jk} \to 0) + a(x_k, y_{jk} \to 0) = \frac{2}{m_{jk}^2}\frac{(1-x_k(1-x_k))^2}{x_k(1-x_k)} \quad \text{(A.117)}$$

with the equivalent limits for $y_{ij} \to 0$ obtained by $i \leftrightarrow k$.

Note 1: the F_3^0 function in GGG is derived from Higgs decay and contains a sum over all three of the permutations of the gluons. Our function corresponds to the sub-antenna function f_3^0 from which it only differs by finite terms. The $f_3^{0'}$ function in the last line is an equivalent reparametrisation of f_3^0 which only differs by terms that cancel when summing over permutations. It is

$$f_3^{0'} = \frac{2y_{ik}}{y_{ij}y_{jk}} + \frac{y_{ik}y_{ij}}{y_{jk}} + \frac{y_{ik}y_{jk}}{y_{ij}} + 2 + y_{ij} + y_{jk}\ . \quad \text{(A.118)}$$

Note 2: the sum of our ++ and – functions is equal to the GGG f_3^0 function modulo a reparametrisation which vanishes when summing over gluon permutations, hence the sum of those radiation functions is equal to F_3^0.

Note 3: the singularity structure of the $gg \to ggg$ radiation function used in ARIADNE differs from ours by terms proportional to $(y_{ik} - y_{ij})/y_{jk}$ and $(y_{ik} - y_{jk})/y_{ij}$, which vanish when summing over neighbouring antennae (they are antisymmetric under interchange of gluons jk and ij respectively). Our parametrisation is chosen to agree with the GGG one which also has the property of minimising the sub-antenna contribution in the hard-collinear limits. (Those configurations are then maximally populated by the soft limit of the neighbouring antenna, which minimises the problem of recoils producing disparate pT scales.)

The individual helicity contributions are:

$$a(++ \to +++) = \frac{1}{s_{IK}} \left[\frac{1}{y_{ij} y_{jk}} \right. \tag{A.119}$$

$$\left. +(1-\alpha) \left((1-y_{ij}) \frac{1-2y_{jk}-y_{ij}}{y_{ij}} + (1-y_{jk}) \frac{1-2y_{ij}-y_{jk}}{y_{jk}} \right) \right]$$

$$a(++ \to +-+) = \frac{1}{s_{IK}} \left[\frac{y_{ik}^3}{y_{ij} y_{jk}} \right] \tag{A.120}$$

$$a(+- \to ++-) = \frac{1}{s_{IK}} \left[\frac{(1-y_{ij})^3}{y_{ij} y_{jk}} + (1-\alpha)(1-y_{ij}) \frac{1-2y_{jk}}{y_{ij}} \right] , \tag{A.121}$$

$$a(+- \to +--) = \frac{1}{s_{IK}} \left[\frac{(1-y_{jk})^3}{y_{ij} y_{jk}} + (1-\alpha)(1-y_{jk}) \frac{1-2y_{ij}}{y_{jk}} \right] . \tag{A.122}$$

Note that the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

A.6.4 GXsplitFF

For the generic case of a massive recoiler, $gX \to q\bar{q}X$, the energy of the parent gluon in the gX rest frame is $E_I = (m_{IK}^2 - m_K^2)/(2m_{IK}) = s_{IK}/2m_{IK}$. The energy fractions of the daughter quarks may then be defined as $x_j = E_j/E_I$ and similarly for x_i, which implies the following relations with the branching invariants,

$$x_j = 1 - y_{ik} \tag{A.123}$$

$$x_i = 1 - y_{jk} , \tag{A.124}$$

This remains valid for generic quark masses, $m_{i,j}$. The helicity-averaged antenna function is:

$$a(g_I X_K \to q_i \bar{q}_j X_k) = \frac{1}{2m_{ij}^2} \left[(1-x_i)^2 + (1-x_j)^2 + \frac{2m_q^2}{m_{ij}^2} \right] \tag{A.125}$$

$$= \frac{1}{2s_{IK}} \frac{1}{y_{ij} + 2\mu_q^2} \left[y_{ik}^2 + y_{jk}^2 + \frac{2\mu_q^2}{y_{ij} + 2\mu_q^2} \right] . \tag{A.126}$$

The individual helicity contributions are:

$$
\begin{aligned}
a(+X \to +-X) &= \frac{1}{2m_{ij}^2}\left[(1-x_j)^2 - \frac{m_q^2}{m_{ij}^2}\frac{(1-x_j)}{x_j}\right] && \text{(A.127)}\\
&= \frac{1}{2s_{IK}}\frac{1}{y_{ij}+2\mu_q^2}\left[y_{ik}^2 - \frac{\mu_q^2}{y_{ij}+2\mu_q^2}\frac{y_{ik}}{1-y_{ik}}\right] , && \text{(A.128)}\\
a(+X \to -+X) &= \frac{1}{2m_{ij}^2}\left[(1-x_i)^2 - \frac{m_q^2}{m_{ij}^2}\frac{(1-x_i)}{x_i}\right] && \text{(A.129)}\\
&= \frac{1}{2s_{IK}}\frac{1}{y_{ij}+2\mu_q^2}\left[y_{jk}^2 - \frac{\mu_q^2}{y_{ij}+2\mu_q^2}\frac{y_{jk}}{1-y_{jk}}\right] , && \text{(A.130)}\\
a(+X \to ++X) &= \frac{1}{2m_{ij}^2}\frac{m_q^2}{m_{ij}^2}\left[\frac{1-x_i}{x_i} + \frac{1-x_j}{x_j} + 2\right] && \text{(A.131)}\\
&= \frac{1}{2s_{IK}}\frac{\mu_q^2}{(y_{ij}+2\mu_q^2)^2}\left[\frac{y_{ik}}{1-y_{ik}} + \frac{y_{jk}}{1-y_{jk}} + 2\right] . && \text{(A.132)}
\end{aligned}
$$

Note that, in the first antenna function, the quark (i) "inherits" the gluon helicity, while in the second one, the antiquark (j) inherits it. The $x \to 1$ limits are suppressed for the x that carries the opposite helicity to that of the splitting gluon. (The third one corresponds to a helicity flip on one of the final-state quarks and is hence proportional to m_q^2.)

A.7 Subleading Colour Corrections

In the strict leading-colour limit, all the gluon-emission antenna functions discussed above are normalized to be proportional to $C_A = N_C = 3$. This will obviously overcount the emission rate from quarks, which should be proportional to $2C_F = 8/3$ (in the same normalisation convention for colour factors that is used throughout VINCIA). In most VINCIA versions until at least v.2.2.02, only the following corrections to this are included in VINCIA:

- $q\bar{q}$ antennae are always proportional to $2C_F$.
- qg antennae are proportional to the average $C_F + C_A/2$, making the error in both the quark and gluon collinear limits of order $1/(2N_C^2) \sim 5\%$.
- Matrix-element corrections (up to the corrected order) are computed at full colour, thus allowing to reabsorb eg the full-colour normalisation of the LEP 4- and 5-jet rates.

In this section, we discuss how further improvements can be implemented, starting from the strict LC approximation.

The treatment is divided into two pieces, one called the "antenna on the back" and the other the "multipole correction". The first of these ensures that collinear radiation from quarks is always proportional to $2C_F$ and also modifies the soft-emission pattern in agreement with known results for double-gluon emission from a $q\bar{q}$ pair. The second is a correction which vanishes in the collinear limits but which introduces further modifications to the soft radiation patterns (essentially from interference between two or more antennae acting coherently), in agreement with known results for $q\bar{q}$ amplitudes with up to four emitted gluons.

A.7.1 The Antenna on the Back

The soft limit of the QCD amplitude for a quark-antiquark pair and two gluons can be represented as a sum over leading-colour eikonal terms (proportional to C_A) minus a "QED-like" eikonal spanned directly between the $q\bar{q}$ pair and proportional to $-1/N_C$ [1]. We note that in directions collinear to the quark or antiquark, this evidently adds up to $2C_F$, as desired. Since this pattern also appears in amplitudes with higher numbers of emitted gluons (but then only as a part of the full subleading-colour corrections, see the section on colour multipoles below), we include it as a generic correction to the radiation from any chain of partons spanned between a quark and antiquark with an arbitrary number of intermediate gluons. Due to the negative sign, it cannot simply be included as an additional probabilistic radiation process (at least not within a positive-definite framework), and it would also not be clear how to represent it in colour space (the most direct interpretation being that a QED-like gluon should be represented as a singlet). Instead, we absorb the correction into the leading-colour pieces, in a manner which guarantees positive-definite radiation functions (at least in the strictly soft limit). We also note that this can be done independently for each colour ordering, since the soft QED-like gluon does not change the relative colour ordering of any hard gluons in the amplitude [1].

A first attempt at this was made in [2], where a simple strictly positive-definite correction factor was defined in complete analogy with VINCIA's matrix-element corrections. In this original approach all of the leading-colour antennae had their radiation strengths reduced slightly,

$$a_i \to a_i R_{\mathrm{NLC}} \tag{A.133}$$

with the subleading-colour correction factor $0 < R_{\mathrm{NLC}} < 1$ defined as:

$$R_{\mathrm{NLC}} = \frac{\sum_{i\in\mathrm{LC}} C_A a_i - \frac{1}{N_C} a_{q\bar{q}}}{\sum_{i\in\mathrm{LC}} C_A a_i} \tag{A.134}$$

This way of absorbing the correction was since abandoned since the explicit sum over all leading-colour contributions required performing all possible $(n+1) \to n$-parton clusterings, which could become extremely time consuming for large parton systems. A trivial exception is the radiation from a $q\bar{q}$ antenna with no intermediate gluons, which just becomes proportional to $2C_F$, hence we just adopt that choice for $q\bar{q}$ antennae and the procedure below is applied to radiation from chains with at least one intermediate gluon.

A less computationally intensive approach is the following. First, the collinear limits of the antenna on the back are absorbed into a redefinition of the qg helicity antenna functions (assuming the LC antennae are normalised to C_A):

$$a_{q_I^+ g_K \to q_i^+ g_j^+ g_k} = a^{\mathrm{LC}}_{q_I^+ g_K \to q_i^+ g_j^+ g_k} - \frac{1}{N_C^2}\frac{1}{m_{IK}^2}\left[\frac{1}{y_{ij}(y_{ij}+y_{jk})}\right] \tag{A.135}$$

$$a_{q_I^+ g_K \to q_i^+ g_j^- g_k} = a^{\mathrm{LC}}_{q_I^+ g_K \to q_i^+ g_j^- g_k} - \frac{1}{N_C^2}\frac{1}{m_{IK}^2}\left[\frac{1}{y_{ij}(y_{ij}+y_{jk})} + \frac{y_{jk}+y_{ij}-2}{y_{ij}}\right] \tag{A.136}$$

A partitioning factor $y_{jk}/(y_{ij}+y_{jk})$ has been applied to the eikonal terms to distribute them among the q and $\bar{q}$ collinear limits, and for the single-pole collinear terms we have used that the collinear splitting fraction $z_i = 1 - y_{jk} = 1 - s_{jk}/m_{IK}^2$ is equal to that of the QED antenna, $z_i' = 1 - s_{j\bar{q}}/s_{qj\bar{q}}$ in the qj collinear limit. This ensures that both the q and $\bar{q}$ collinear limits will have the correct colour factor.

The pure soft structure of the antenna on the back is represented as a pure eikonal proportional to $\frac{1}{p_{\perp\mathrm{QED}}^2}$. This is absorbed by identifying the LC sector with the smallest $p_\perp^2$ scale and subtracting the following correction, which takes into account that the qg and $g\bar{q}$ antennae have already absorbed their parts:

$$a_{\mathrm{hel}} = a_{\mathrm{hel}}^{\mathrm{LC}} + \frac{1}{N_C^2}\left[\frac{1}{p_{\perp 1}^2}\frac{s_{12}}{s_{q1}+s_{12}} + \frac{1}{p_{\perp n}^2}\frac{s_{n-1,n}}{s_{n-1,n}+s_{n\bar{q}}} - \frac{1}{p_{\perp\mathrm{QED}}^2}\right] \tag{A.137}$$

where the (colour-ordered) gluons are indexed from 1 to n (so the quark would have index 0 and the antiquark index $n+1$). The "hel" subscript emphasises that this correction is added to each of the helicity antennae, so that antenna functions where the helicity of the emitted gluon is summed over will have twice as large a correction.

A.7.2 The Colour-Multipole Correction

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A.8 Choice of (Re)start Scale for Next Trial

The trial Sudakov factor is defined as:

$$\hat{\Delta}(Q_1^2, Q_2^2) = \exp\left[-\hat{\mathcal{A}}(Q_1^2, Q_2^2)\right] \ , \tag{A.138}$$

and the next trial scale is found by solving the equation:

$$\mathcal{R} = \hat{\Delta}(Q^2, Q^2_{\mathrm{new}}) \ , \tag{A.139}$$

for Q_{new}, with $\mathcal{R}$ a random number distributed uniformly in the interval $\mathcal{R} \in [0,1]$, and Q the current "restart scale". For strongly ordered showers, the restart scale after an accepted trial branching is the evolution scale of the last generated branching. For smoothly ordered showers, this restart scale is only used for antennae that are not color-adjacent to the branching that occurred; for the newly created antennae, and (optionally) for any color-adjacent ones, the restart scale is the respective antenna invariant masses[2].

For both strongly and smoothly ordered showers, the restart scale after a failed (vetoed) trial branching is the scale of the failed branching.

Note: to optimize event generation, trial scales can be saved and reused for any antennae whose flavors, spins, and invariant masses are preserved by the preceding branching step.

B Initial–Initial Evolution Equations

B.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $AB \to ajb$, respectively, for initial-state partons A and B evolving backwards to partons a, j, and b, with j in the final state and a and b in the initial state. (From the perspective of forwards evolution, partons a and b emit parton j.) For branchings involving initial-state partons, there is the additional aspect of the "hard system", which we denote R (for "recoiler") and which in general experiences a frame reinterpretation (Lorentz transformation = rotation + boost) as a consequence of the branching. Thus, the pre-branching system is $AB \to R$, with momentum conservation implying $p_A + p_B = p_R$, and the post-branching system is $ab \to j + r$ with $p_a + p_b - p_j = p_r$. For general masses, and using the notation $s_{12} \equiv 2p_1 \cdot p_2$, conservation of the invariant mass of the hard system ($m_R = m_r$) implies the relations

$$m_{ab}^2 \equiv s_{ab} + m_a^2 + m_b^2 = s_{AB} + s_{aj} + s_{jb} + m_A^2 + m_B^2 - m_j^2 \,, \tag{B.1}$$

$$s_{AB} + m_A^2 + m_B^2 = s_{ab} - s_{aj} - s_{jb} + m_a^2 + m_b^2 + m_j^2 \,. \tag{B.2}$$

For dimensionless equivalents, we normalise by the largest invariant, s_{ab}, hence for example

$$1 + \mu_a^2 + \mu_b^2 = y_{AB} + y_{aj} + y_{jb} + \mu_A^2 + \mu_B^2 - \mu_j^2 \,. \tag{B.3}$$

When a gluon is emitted into the final state we have $m_A = m_a$, $m_B = m_b$, and $m_j = 0$, hence

$$\text{Final-state gluon emission:} \quad s_{ab} - s_{aj} - s_{jb} = s_{AB} \,. \tag{B.4}$$

For quark creation (a.k.a. quark conversion: a quark backwards evolving to a gluon) on side a, we have $m_a = 0$, $m_A = m_q$, $m_j = m_q$, and $m_B = m_b$, hence also

$$g_a \to q_A \bar{q}_j: \quad s_{ab} - s_{aj} - s_{jb} = s_{AB} \,, \tag{B.5}$$

For gluon creation (aka gluon conversion; a gluon backwards evolving to a quark) on side a, we have $m_a = m_q$, $m_A = 0$, $m_j = m_q$, and $m_B = m_b$, hence

$$q_a \to g_A q_j: \quad s_{ab} - s_{aj} - s_{jb} + 2m_q^2 = s_{AB} \,. \tag{B.6}$$

Note that, although the relations above have been expressed for branchings with general masses, the current initial-state shower implementation assumes incoming legs to be explicitly massless.

[2]This allows hard $2 \to n$ branchings to be generated inside the newly created antennae (and optionally within the color-adjacent ones) without disturbing the evolution of the rest of the event.

B.2 Crossing Relations

Compared to the FF case, crossing symmetry for the case when the recoiling partons i and k are crossed to be identified with a and b, respectively, implies

$$y_{ik} \quad \to \quad \frac{s_{ab}}{s_{AB}} \equiv 1/z \tag{B.7}$$

$$y_{ij} \quad \to \quad -\frac{s_{aj}}{s_{AB}} \equiv -y_{aj}/z \tag{B.8}$$

$$y_{jk} \quad \to \quad -\frac{s_{jb}}{s_{AB}} \equiv -y_{jb}/z \ . \tag{B.9}$$

For crossings of partons $i \to a$ and $j \to b$,

$$y_{ik} \quad \to \quad -\frac{s_{ak}}{s_{AB}} \equiv -y_{ak}/z \tag{B.10}$$

$$y_{ij} \quad \to \quad \frac{s_{ab}}{s_{AB}} \equiv 1/z \tag{B.11}$$

$$y_{jk} \quad \to \quad -\frac{s_{kb}}{s_{AB}} \equiv -y_{kb}/z \ , \tag{B.12}$$

where, if desired, one can obviously do a relabeling $k \to j$ for the II antennae such that the parton that remains in the final state is still labeled j.

For crossings of partons $j \to a$ and $k \to b$,

$$y_{ik} \quad \to \quad -\frac{s_{ib}}{s_{AB}} \equiv -y_{ib}/z \tag{B.13}$$

$$y_{ij} \quad \to \quad -\frac{s_{ai}}{s_{AB}} \equiv -y_{ai}/z \tag{B.14}$$

$$y_{jk} \quad \to \quad \frac{s_{ab}}{s_{AB}} \equiv 1/z \ . \tag{B.15}$$

Note also that outgoing particles are mapped to incoming particles with the opposite helicities, so that, e.g., the II antenna function for $(\bar{q}_A^+ q_B^+ \to \bar{q}_a^+ g_j^+ q_b^+)$ is obtained from the FF one for $(q_I^- \bar{q}_K^- \to q_i^- g_j^+ \bar{q}_k^-)$.

Finally, we note that, in the current implementation of initial-state antenna functions, there are no II antenna functions for "emission into the initial state"; specifically, when both partons j and k are gluons, the crossing of ij is not used and similarly when both i and j are gluons. Taking the FF function for $qg \to qgg$ as an example, the function resulting from crossing quark i and gluon j is not used. (Instead, those terms are associated with emissions in the IF antenna between parton b and the recoiler, which produces the same colour structure). This is a choice of convention which effectively corresponds to a partial sectoring of the initial-state shower. It could be changed if future directions warrant it. Presently, the colour chain $a - b - j - R$ only contains the (IF) clustering of j into b and R but not one corresponding to "clustering" b into a

and j (hence the name “emission into the initial state”). The abj antenna function and clustering could still be envisioned (with a corresponding subtraction of the terms present in the current bjR one); it would use an II kinematics map, possibly with fixed x_a. The choice of which mapping is associated with these terms affects whether they generate recoils (II) or not (IF). It could be an interesting student project to expand to “emissions into the initial state” and see how this affects recoil distributions, possibly in association with using an II map for the initial part of IF antennae.

B.3 Antenna Functions

B.3.1 QQemitII

The helicity-averaged antenna function is:

$$a(\bar{q}_A q_B \to \bar{q}_a g_j q_b) = \frac{1}{s_{AB}}\left[\frac{2y_{AB}}{y_{aj}y_{jb}} + \frac{y_{jb}}{y_{aj}} + \frac{y_{aj}}{y_{jb}} + 1 - \frac{2\mu_a^2(1-y_{jb})}{y_{aj}^2} - \frac{2\mu_b^2(1-y_{aj})}{y_{jb}^2}\right] \tag{B.16}$$

$$= \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^2+(1-y_{jb})^2}{y_{aj}y_{jb}} + 1 - \frac{2\mu_a^2(1-y_{jb})}{y_{aj}^2} - \frac{2\mu_b^2(1-y_{aj})}{y_{jb}^2}\right] . \tag{B.17}$$

The collinear limit of this antenna function is:

$$a(y_{aj} \to 0, y_{jb} \to 1-z) = \frac{1}{z}\frac{1}{s_{aj}}\left[\frac{1+z^2}{1-z} - \frac{2zm_a^2}{s_{aj}}\right] \tag{B.18}$$

For comparison to standard DGLAP splitting kernels, note that $s_{aj} = 2p_a \cdot p_j = -((p_a - p_j)^2 - m_a^2) = -(p_a - p_j)^2 + m_a^2 \equiv Q_A^2 + m_a^2$.

The individual helicity contributions are:

$$a(++\to+++) = \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}} - \frac{\mu_a^2}{y_{aj}^2} - \frac{\mu_b^2}{y_{jb}^2}\right], \quad (B.19)$$

$$a(++\to+-+) = \frac{1}{s_{AB}}\left[\frac{y_{AB}^2}{y_{aj}y_{jb}} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2} - \frac{\mu_b^2(1-y_{aj})^2}{y_{jb}^2}\right], \quad (B.20)$$

$$a(++\to--+) = \frac{1}{s_{AB}}\left[\frac{\mu_a^2 y_{jb}^2}{y_{aj}^2}\right], \quad (B.21)$$

$$a(++\to+--) = \frac{1}{s_{AB}}\left[\frac{\mu_b^2 y_{aj}^2}{y_{jb}^2}\right], \quad (B.22)$$

$$a(+-\to++-) = \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^2}{y_{aj}y_{jb}} - \frac{\mu_a^2}{y_{aj}^2} - \frac{\mu_b^2(1-y_{aj})^2}{y_{jb}^2}\right], \quad (B.23)$$

$$a(+-\to+--) = \frac{1}{s_{AB}}\left[\frac{(1-y_{jb})^2}{y_{aj}y_{jb}} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2} - \frac{\mu_b^2}{y_{jb}^2}\right], \quad (B.24)$$

$$a(+-\to---) = \frac{1}{s_{AB}}\left[\frac{\mu_a^2 y_{jb}^2}{y_{aj}^2}\right], \quad (B.25)$$

$$a(+-\to-++) = \frac{1}{s_{AB}}\left[\frac{\mu_b^2 y_{aj}^2}{y_{jb}^2}\right]. \quad (B.26)$$

Note that the sum of the ++ antenna functions has the same singularities as the sum of the +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

B.3.2 QGemitII

The helicity-averaged antenna function is:

$$a(q_A g_B \to q_a g_j g_b) = \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^3+(1-y_{jb})^2}{y_{aj}y_{jb}} + \frac{1+y_{aj}^3}{y_{jb}(1-y_{aj})} - \frac{2\mu_a^2(1-y_{jb})}{y_{aj}^2} + 2 - y_{aj} - \frac{y_{jb}}{2}\right]. \quad (B.27)$$

Note that the singular structure differs from the corresponding FF form by the term proportional to $1/(1-y_{aj})$ which is a "sector" term (necessary since VINCIA does not include a sector for emission into the initial state). For comparison, the corresponding FF antenna is of the global type, and includes an antisymmetric $j \leftrightarrow k$ term which cancels when summing over neighbouring antennae.

The collinear limits are:

$$a(y_{aj} \to 0, y_{jb} \to 1-z) \;=\; \frac{1}{z}\frac{1}{s_{aj}}\left[\frac{1+z^2}{1-z} - \frac{2zm_a^2}{s_{aj}}\right] , \tag{B.28}$$

$$a(y_{aj} \to 1-z, y_{jb} \to 0) \;=\; \frac{1}{z}\frac{1}{s_{jb}}\frac{2(1+z(1-z))^2}{z(1-z)} \tag{B.29}$$

For comparison to standard DGLAP splitting kernels, note that $s_{aj} = 2p_a \cdot p_j = -((p_a - p_j)^2 - m_a^2) = -(p_a - p_j)^2 + m_a^2 \equiv Q_A^2 + m_a^2$.

The individual helicity contributions are crossings of FF sector antennae (see LLS):

$$a(++ \to +++) \;=\; \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}}\frac{1-y_{jb}}{1-y_{aj}-y_{jb}} - \frac{\mu_a^2}{y_{aj}^2}\right] \tag{B.30}$$

$$= \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}y_{AB}} - \frac{\mu_a^2}{y_{aj}^2}\right] \tag{B.31}$$

$$\overset{\text{sing}}{\to} \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}(1-y_{aj})} - \frac{\mu_a^2}{y_{aj}^2}\right] . \tag{B.32}$$

$$a(++ \to +-+) \;=\; \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}}\frac{y_{AB}^3}{1-y_{jb}} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2}\right] \tag{B.33}$$

$$= \frac{y_{AB}^3}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{aj}(1-y_{jb})} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2}\right] \tag{B.34}$$

$$\overset{\text{sing}}{\to} \frac{1}{s_{AB}}\left[\frac{y_{AB}^3}{y_{aj}y_{jb}} + \frac{y_{AB}^2}{y_{aj}} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2}\right] \tag{B.35}$$

$$= \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})y_{AB}^2}{y_{aj}y_{jb}} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2}\right] , \tag{B.36}$$

$$a(++ \to --+) \;=\; \frac{1}{s_{AB}}\left[\frac{\mu_a^2 y_{jb}^2}{y_{aj}^2}\right] , \tag{B.37}$$

$$a(++ \to +--) \;=\; \frac{1}{s_{AB}}\left[\frac{y_{aj}^3}{y_{jb}(1-y_{jb})}\frac{1}{1-y_{aj}-y_{jb}}\right] \tag{B.38}$$

$$\overset{\text{sing}}{\to} \frac{1}{s_{AB}}\frac{y_{aj}^3}{y_{jb}(1-y_{aj})} , \tag{B.39}$$

$$\tag{B.40}$$

$$a(+- \to ++-) = \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^3}{y_{aj}y_{jb}} + \frac{1-y_{jb}-y_{aj}^2}{1-y_{jb}} - \frac{\mu_a^2(1-y_{aj})}{y_{aj}^2}\right] \tag{B.41}$$

$$\stackrel{\text{sing}}{\to} \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^3}{y_{aj}y_{jb}} - \frac{\mu_a^2}{y_{aj}^2}\right] , \tag{B.42}$$

$$a(+- \to +--) = \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}}\frac{(1-y_{jb})^3}{1-y_{aj}-y_{jb}} - \frac{\mu_a^2(1-y_{aj}-y_{jb})^2}{y_{aj}^2(1-y_{aj})}\right] \tag{B.43}$$

$$= \frac{(1-y_{jb})^2}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}}\frac{1}{1-y_{aj}-y_{jb}} - \frac{\mu_a^2}{y_{aj}^2}\frac{(1-y_{aj}-y_{jb})^2}{(1-y_{jb})^2(1-y_{aj})}\right]$$

$$\stackrel{\text{sing}}{\to} \frac{1}{s_{AB}}\left[\frac{(1-y_{jb})^2}{y_{aj}y_{jb}} + \frac{1}{y_{jb}(1-y_{aj})} - \frac{\mu_a^2(1-y_{jb})^2}{y_{aj}^2}\right] , \tag{B.44}$$

$$a(+- \to ---) = a(++ \to --+) , \tag{B.45}$$

$$a(+- \to +++) = a(++ \to +--) . \tag{B.46}$$

Note that, unlike the case for the global (FF) antennae, there are also non-zero sector antenna functions involving helicity flips of the parent gluons; in a global language, these terms would be generated by the neighbouring antenna, but since VINCIA currently does not include a sector for "emission into the initial state", the terms for which the helicity of incoming parton b is imparted to the outgoing gluon j, with B having positive helicity, must be included, leading to the additional sector terms that appear in the functions above (proportional to $1/(y_{jb}(1-y_{aj})) \sim 1/(y_{jb}z)$).

Note: the $(1-y_{jb})$ and $(1-y_{aj})$ denominators are approximations of $1/y_{AB}$ denominators from the crossing. The corresponding $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$). The forms with $1/y_{AB}$ may still be useful for trial overestimates, since $y_{AB} = z$ over all of our phase space.

B.3.3 GGemitII

The helicity-averaged antenna function is:

$$a(g_A g_B \to g_a g_j g_b) = \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^3+(1-y_{jb})^3}{y_{aj}y_{jb}} + \frac{1+y_{aj}^3}{y_{jb}(1-y_{aj})} + \frac{1+y_{jb}^3}{y_{aj}(1-y_{jb})} + 3 - \frac{3y_{aj}}{2} - \frac{3y_{jb}}{2}\right] . \tag{B.47}$$

Note that this form is equivalent to the corresponding FF one, up to the terms proportional to $1/(1-y_{aj})$ and $1/(1-y_{jb})$ which are "sector" terms (necessary since VINCIA does not include a sector for emission into the initial state). For comparison, the corresponding FF antenna is of the global type, and includes an antisymmetric $j \leftrightarrow k$ term which cancels when summing over neighbouring antennae.

The collinear limits are:

$$a(y_{aj} \to 0, y_{jb} \to 1-z) = \frac{1}{z}\frac{1}{s_{aj}}\frac{2(1+z(1-z))^2}{z(1-z)}\,, \tag{B.48}$$

$$a(y_{aj} \to 1-z, y_{jb} \to 0) = \frac{1}{z}\frac{1}{s_{jb}}\frac{2(1+z(1-z))^2}{z(1-z)} \tag{B.49}$$

The individual helicity contributions are crossings of FF sector antennae (see LLS and the comments and limits taken in the QGEmitII section; here we just give the final forms):

$$a(++\to+++) = \frac{1}{s_{AB}}\left[\frac{1}{y_{aj}y_{jb}}+\frac{1}{y_{jb}(1-y_{aj})}+\frac{1}{y_{aj}(1-y_{jb})}\right]\,, \tag{B.50}$$

$$a(++\to+-+) = \frac{1}{s_{AB}}\frac{y_{AB}^3}{y_{aj}y_{jb}}\,, \tag{B.51}$$

$$a(+-\to++-) = \frac{1}{s_{AB}}\left[\frac{(1-y_{aj})^3}{y_{aj}y_{jb}}+\frac{1}{y_{aj}(1-y_{jb})}\right]\,, \tag{B.52}$$

$$a(+-\to+--) = \frac{1}{s_{AB}}\left[\frac{(1-y_{jb})^3}{y_{aj}y_{jb}}+\frac{1}{y_{jb}(1-y_{aj})}\right] \tag{B.53}$$

$$\tag{B.54}$$

The nonzero functions involving helicity flips of the parent gluons (see the comments in the QGEmitII section) are:

$$a(++\to+--) = \frac{1}{s_{AB}}\frac{y_{aj}^3}{y_{jb}(1-y_{aj})}\,, \tag{B.55}$$

$$a(++\to--+) = \frac{1}{s_{AB}}\frac{y_{jb}^3}{y_{aj}(1-y_{jb})}\,, \tag{B.56}$$

$$a(+-\to+++) = a(++\to+--)\,, \tag{B.57}$$

$$a(+-\to---) = a(++\to--+)\,. \tag{B.58}$$

Note: the $(1-y_{jb})$ and $(1-y_{aj})$ denominators are approximations of $1/y_{AB}$ denominators from the crossing. The corresponding $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$). The difference are the terms that are resummed by programs like HEJ.

B.3.4 QXsplitII

Quark (or antiquark) in the initial state backwards-evolving into a gluon and emitting an antiquark (quark) into the final state. Can be obtained by crossing the QQemitFF functions, with final-state parton j as one of the crossed partons and relabeling. (Note hence this function does provide an example of a crossing that corresponds to "emission into the initial state".)

The helicity-averaged antenna functiuon is:

$$a(q_A X_B \to g_a \bar{q}_j X_b) = \frac{1}{s_{AB}} \left[\frac{y_{AB}^2 + (1-y_{AB})^2}{y_{aj}} + \frac{2\mu_j^2 y_{AB}}{y_{aj}^2} \right] \tag{B.59}$$

$$= \frac{1}{z} \frac{1}{s_{aj}} \left[z^2 + (1-z)^2 + \frac{2zm_j^2}{s_{aj}} \right] \tag{B.60}$$

The individual helicity contributions are:

$$a(+X \to +-X) = \frac{1}{s_{AB}} \left[\frac{y_{AB}^2}{y_{aj}} - \frac{\mu_j^2 y_{AB}^2}{y_{aj}^2(1-y_{AB})} \right] , \tag{B.61}$$

$$a(+X \to --X) = \frac{1}{s_{AB}} \left[\frac{(1-y_{AB})^2}{y_{aj}} - \frac{\mu_j^2(1-y_{AB})}{y_{aj}^2} \right] , \tag{B.62}$$

$$a(+X \to ++X = \frac{1}{s_{AB}} \left[\frac{\mu_j^2}{y_{aj}^2(1-y_{AB})} \right] , \tag{B.63}$$

$$a(-X \to -+X) = a(+X \to +-X) , \tag{B.64}$$

$$a(-X \to ++X) = a(+X \to --X) , \tag{B.65}$$

$$a(-X \to --X) = a(+X \to ++X) . \tag{B.66}$$

Note that the numerators express helicity conservation. The same expressions hold for backwards evolution of an antiquark, i.e. for $\bar{q}_A X_B$.

Note for comparison to standard DGLAP kernels that $s_{aj} \equiv 2p_a \cdot p_j = -((p_a - p_j)^2 - m_j^2) = -(p_a - p_j)^2 + m_j^2 \equiv Q_A^2 + m_j^2$, since $m_a = m_g = 0$.

B.3.5 GXconvII

Gluon in the initial state backwards-evolving into a quark (or antiquark) and emitting a quark (antiquark) into the final state. These functions can be obtained by crossing either parton i or j in the GXsplitFF functions.

The helicity-averaged antenna function (for general $m_a = m_j$) is:

$$a(g_A X_B \to q_a q_j X_b) = \frac{1}{2s_{AB}} \left[\frac{1 + (1-y_{AB})^2}{(y_{aj} - 2\mu_j^2) y_{AB}} - \frac{2\mu_j^2 y_{AB}}{(y_{aj} - 2\mu_j^2)^2} \right] . \tag{B.67}$$

The individual helicity contributions are:

$$a(+X \to ++X) = \frac{1}{2s_{AB}}\left[\frac{1}{(y_{aj}-2\mu_j^2)y_{AB}} - \frac{\mu_j^2}{(y_{aj}-2\mu_j^2)^2}\frac{y_{AB}}{1-y_{AB}}\right] , \quad \text{(B.68)}$$

$$a(+X \to --X) = \frac{1}{s_{AB}}\left[\frac{(1-y_{AB})^2}{(y_{aj}-2\mu_j^2)y_{AB}} - \frac{\mu_j^2 y_{AB}(1-y_{AB})}{(y_{aj}-2\mu_j^2)^2}\right] , \quad \text{(B.69)}$$

$$a(+X \to +-X) = \frac{1}{s_{AB}}\left[\frac{\mu_j^2}{(y_{aj}-2\mu_j^2)^2}\frac{y_{AB}^3}{1-y_{AB}}\right] , \quad \text{(B.70)}$$

$$a(-X \to --X) = a(+X \to ++X) , \quad \text{(B.71)}$$

$$a(-X \to ++X) = a(+X \to --X) , \quad \text{(B.72)}$$

$$a(-X \to -+X) = a(+X \to +-X) . \quad \text{(B.73)}$$

Note that the numerators express helicity conservation. The same expressions hold for backwards to an antiquark, i.e. for $g_A X_B \to \bar{q}_a \bar{q}_j X_b$.

Note 2: the $1/y_{AB}$ denominators could be approximated by $1/(1-y_{jb})$ since the $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$).

B.4 Evolution Variables

The evolution variables we use are

$$Q_\perp^2 = \frac{s_{aj}s_{jb}}{s_{ab}} = \frac{s_{aj}s_{jb}}{s_{AB}+s_{aj}+s_{jb}} , \quad \text{(B.74)}$$

$$Q_A^2 = s_{aj} , \quad \text{(B.75)}$$

$$Q_B^2 = s_{jb} . \quad \text{(B.76)}$$

Phase Space Boundaries: With $s_{ab} = s_{AB} + s_{aj} + s_{jb}$ and $s_{ab} \leq s$, the phase space boundaries are $0 \leq s_{aj} + s_{jb} \leq s - s_{AB}$ and for the single branching invariants $0 \leq s_{aj} \leq s - s_{AB}$ and $0 \leq s_{jb} \leq s - s_{AB}$. The maxima of the evolution variables are

$$Q_{\perp\ \mathrm{max}}^2 = \frac{1}{4}\frac{(s-s_{AB})^2}{s} , \quad \text{(B.77)}$$

$$Q_{A\ \mathrm{max}}^2 = s - s_{AB} , \quad \text{(B.78)}$$

$$Q_{B\ \mathrm{max}}^2 = s - s_{AB} . \quad \text{(B.79)}$$

B.5 Zeta Definitions

The choices are

$$\zeta_1 = \frac{s_{aj}}{s_{ab}} = \frac{s_{aj}}{s_{AB} + s_{aj} + s_{jb}} = y_{aj} \ , \tag{B.80}$$

$$\zeta_2 = \frac{s_{aj}}{s_{AB}} = \frac{s_{ab}}{s_{AB}} y_{aj} \ , \tag{B.81}$$

$$\zeta_3 = \frac{s_{jb}}{s_{AB}} \ , \tag{B.82}$$

$$\zeta_4 = \frac{s_{ab}}{s_{AB}} \ . \tag{B.83}$$

The integration boundaries for the ζ variables are

$$\zeta_{1\pm}(Q_\perp^2) = \frac{1}{2s}\left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_\perp^2 s}\right) \ , \tag{B.84}$$

$$\zeta_{2\pm}(Q_\perp^2) = \frac{1}{2s_{AB}}\left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_\perp^2 s}\right) \ , \tag{B.85}$$

$$\zeta_{3\pm}(Q_\perp^2) = \frac{1}{2s_{AB}}\left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_\perp^2 s}\right) \ , \tag{B.86}$$

$$\zeta_{4-}(Q_{A/B}^2) = \frac{s_{AB} + Q_{A/B}^2}{s_{AB}} \qquad \zeta_{4+}(Q_{A/B}^2) = \frac{s}{s_{AB}} \ . \tag{B.87}$$

B.6 Jacobians

The Jacobians for the transformation from the phase-space variables, (s_{aj}, s_{jb}), to the shower variables, (Q_E, ζ), are

$$|J(Q_\perp^2, \zeta_1)| = s_{ab}\frac{1}{\zeta_1(1-\zeta_1)} \ , \tag{B.88}$$

$$|J(Q_\perp^2, \zeta_2)| = s_{ab}^2 \frac{1}{s_{aj}} \frac{1}{1+\zeta_2} \ , \tag{B.89}$$

$$|J(Q_\perp^2, \zeta_3)| = s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1+\zeta_3} \ , \tag{B.90}$$

$$|J(Q_A^2, \zeta_4)| = s_{AB} \ , \tag{B.91}$$

$$|J(Q_B^2, \zeta_4)| = s_{AB} \ . \tag{B.92}$$

B.7 Trial Functions

The following II trial functions are used:

$$\hat{a}_{\text{soft}} = \frac{1}{s_{AB}}\frac{2s_{ab}^2}{s_{aj}s_{jb}} \quad = \frac{1}{s_{AB}}\frac{2}{y_{aj}y_{jb}} \,, \tag{B.93}$$

$$\hat{a}_{\text{coll A}} = 2\frac{s_{ab}^2}{s_{AB}^2}\frac{1}{s_{aj}} \quad = \frac{1}{s_{AB}}\frac{2}{y_{aj}(1-y_{aj}-y_{jb})} \,, \tag{B.94}$$

$$\hat{a}_{\text{coll B}} = 2\frac{s_{ab}^2}{s_{AB}^2}\frac{1}{s_{jb}} \quad = \frac{1}{s_{AB}}\frac{2}{y_{jb}(1-y_{aj}-y_{jb})} \,, \tag{B.95}$$

$$a_{\text{split A}} = \frac{1}{s_{AB}}\left(-2\frac{s_{jb}s_{AB}}{s_{aj}(s_{ab}-s_{aj})} + \frac{s_{ab}}{s_{aj}}\right) \quad \Rightarrow \quad \hat{a}_{\text{split A}} = \frac{s_{ab}}{s_{AB}}\frac{1}{s_{aj}} \,, \tag{B.96}$$

$$a_{\text{split B}} = \frac{1}{s_{AB}}\left(-2\frac{s_{aj}s_{AB}}{s_{jb}(s_{ab}-s_{jb})} + \frac{s_{ab}}{s_{jb}}\right) \quad \Rightarrow \quad \hat{a}_{\text{split B}} = \frac{s_{ab}}{s_{AB}}\frac{1}{s_{jb}} \,, \tag{B.97}$$

$$a_{\text{conv A}} = \frac{1}{2s_{aj}}\frac{s_{jb}^2+s_{ab}^2}{s_{AB}^2} \quad \Rightarrow \quad \hat{a}_{\text{conv A}} = \frac{s_{ab}^2}{s_{AB}^2}\frac{1}{s_{aj}} \,, \tag{B.98}$$

$$a_{\text{conv B}} = \frac{1}{2s_{jb}}\frac{s_{aj}^2+s_{ab}^2}{s_{AB}^2} \quad \Rightarrow \quad \hat{a}_{\text{conv B}} = \frac{s_{ab}^2}{s_{AB}^2}\frac{1}{s_{jb}} \,. \tag{B.99}$$

Note that the overestimate of the soft eikonal term is simultaneously an overestimate of the collinear singularities for quarks, so the extra collinear trial functions are only needed for the additional ($1/x$-enhanced) terms that appear in initial-state $g \to gg$ branchings due to the missing sector for "emission into the initial state".

B.8 PDF Ratios

We define the trial PDF ratio:

$$\hat{R}_f = \frac{f_a(x_A, Q_A^2)f_b(x_B, Q_B^2)}{f_A(x_A, Q_A^2)f_B(x_B, Q_B^2)} \tag{B.100}$$

for the PDF ratio evaluated at the pre-branching x and Q^2 values (with Q^2 sliding to be the starting scale for the current trial). Note that this is unity for gluon emissions (when $a = A$ and $b = B$) and reduces to a single ratio when either $a = A$ or $b = B$. Technically, even when this is unity, a number different from one may be coded in the corresponding method, as a kind of hardcoded headroom factor that can compensate for (small) PDF ratio overestimate excesses, so strictly speaking we define:

$$\hat{R}_f = k\frac{f_a(x_A, Q_A^2)f_b(x_B, Q_B^2)}{f_A(x_A, Q_A^2)f_B(x_B, Q_B^2)} \,, \tag{B.101}$$

with k a constant of order unity which can be chosen differently for different trials.

In many cases, the PDFs fall off at least as $1/x$ towards higher x, so that the physical PDF ratio can be overestimated by

$$R_f \equiv \frac{f_a(x_a, Q_a^2) f_b(x_b, Q_b^2)}{f_A(x_A, Q_A^2) f_B(x_B, Q_B^2)} \leq \frac{x_A x_B}{x_a x_B} \hat{R}_f = \frac{s_{AB}}{s_{ab}} \hat{R}_f \tag{B.102}$$

Thus, in most trial integrals below, the above substitution, $R_f \to (s_{AB}/s_{ab})\hat{R}_f$ is made, and the accept ratio will then contain a factor

$$P_{\rm PDF}^{\rm accept} = \frac{s_{ab}}{s_{AB}} \frac{R_f}{\hat{R}_f} \equiv \frac{R_{xf}}{\hat{R}_f} \ . \tag{B.103}$$

For some trial generators, however, we either do not wish to make this overestimate (e.g., valence quark distributions) or have not managed to find a simple enough phase-space parametrisation to work with, so that we instead have to use the larger overestimate

$$R_f \leq \hat{R}_f \ . \tag{B.104}$$

When this is the case, we modify the trial-function method in the code to return $(s_{ab}/s_{AB})\hat{a}$ so that the code can still use the above accept ratio without having to worry about which type of overestimate was used to generate the trial. (The two fators s_{ab}/s_{AB} then cancel in the antenna$\times$PDF trial ratio.) Note that this approach remains valid when mixing / combining trial generators that use different PDF overestimates.

B.9 Integration Kernels

Using the default $xf \leq \mathrm{const}$ overestimate, the overestimate of the evolution integral is

$$\hat{A}_{xf}(Q_{E\,\rm F}^2, Q_{E\,\rm new}^2) = \int_{Q_{E\,\rm new}^2}^{Q_{E\,\rm F}^2} \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}^2}{s_{ab}^3} \, \hat{a} \, \hat{R}_f \, |J| \, \mathrm{d}Q_E^2 \mathrm{d}\zeta \, \frac{\mathrm{d}\phi}{2\pi} \ . \tag{B.105}$$

When using the number-density overestimate, we have instead

$$\hat{A}_{f}(Q_{E\,\rm F}^2, Q_{E\,\rm new}^2) = \int_{Q_{E\,\rm new}^2}^{Q_{E\,\rm F}^2} \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \, \hat{a} \, \hat{R}_f \, |J| \, \mathrm{d}Q_E^2 \mathrm{d}\zeta \, \frac{\mathrm{d}\phi}{2\pi} \ . \tag{B.106}$$

The integration kernels for the Q_E^2 integration are, for xf overestimates:

1. Soft with $Q_\perp^2$: none.
2. A gluon collinear with $Q_\perp^2$ and ζ_3 (similar B gluon collinear with ζ_2):

$$\begin{aligned} \mathrm{d}\hat{\mathcal{A}}_{\rm coll\,A}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}^2}{s_{ab}^3} \, 2 \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} \, s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1+\zeta_3} \, \hat{R}_f \, \mathrm{d}Q_\perp^2 \mathrm{d}\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \frac{\mathrm{d}\zeta_3}{1+\zeta_3} \end{aligned} \tag{B.107}$$

3. A gluon splitting
   (a) with $Q_\perp^2$: none.
   (b) with Q_A^2 and ζ_4 (similar B gluon splitting with Q_B^2 and ζ_4):

$$\mathrm{d}\hat{\mathcal{A}}_{\text{split A}}(Q_A^2) = \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \frac{\mathrm{d}\zeta_4}{\zeta_4^2} \tag{B.108}$$

4. A conversion (note this overestimate would only be appropriate for converesion back to sea quarks; valence quarks cannot be assumed to fall like $1/x$).
   (a) with $Q_\perp^2$ and ζ_3 (similar B conversion with ζ_2):

$$\mathrm{d}\hat{\mathcal{A}}_{\text{conv A}}(Q_\perp^2) = \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \frac{\mathrm{d}\zeta_3}{1+\zeta_3} \tag{B.109}$$

   (b) with Q_A^2 and ζ_4 (similar B conversion with Q_B^2 and ζ_4):

$$\mathrm{d}\hat{\mathcal{A}}_{\text{conv A}}(Q_A^2) = \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \frac{\mathrm{d}\zeta_4}{\zeta_4} \tag{B.110}$$

When using number-density overestimates instead of xf ones, the following integration kernels can be used:

1. Soft with $Q_\perp^2$ and ζ_1:

$$\begin{aligned}\mathrm{d}\hat{\mathcal{A}}_{\text{soft}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{1}{s_{AB}} \frac{2s_{ab}^2}{s_{aj}s_{jb}}\, s_{ab} \frac{1}{\zeta_1(1-\zeta_1)}\, \hat{R}_f\, \mathrm{d}Q_\perp^2 \mathrm{d}\zeta_1 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \frac{\mathrm{d}\zeta_1}{\zeta_1(1-\zeta_1)}\end{aligned} \tag{B.111}$$

2. A gluon collinear with $Q_\perp^2$: none.

3. A gluon splitting
   (a) with $Q_\perp^2$ and ζ_3 (similar B gluon splitting with ζ_2):

$$\begin{aligned}\mathrm{d}\hat{\mathcal{A}}_{\text{split A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}}{s_{AB}} \frac{1}{s_{aj}}\, s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1+\zeta_3}\, \hat{R}_f\, \mathrm{d}Q_\perp^2 \mathrm{d}\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \frac{\mathrm{d}\zeta_3}{1+\zeta_3}\end{aligned} \tag{B.112}$$

(b) with Q_A^2 and ζ_4 (similar B gluon splitting with Q_B^2 and ζ_4):

$$
\begin{aligned}
\mathrm{d}\hat{\mathcal{A}}_{\text{split A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}}{s_{AB}} \frac{1}{s_{aj}} \, s_{AB} \, \hat{R}_f \, \mathrm{d}Q_A^2 \mathrm{d}\zeta_4 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{s_{AB}}{s_{ab}} \frac{\mathrm{d}Q_A^2}{Q_A^2} \mathrm{d}\zeta_4 \left(\frac{x_a x_b}{x_A x_B} \frac{1}{\zeta_4} \right) \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \frac{\mathrm{d}\zeta_4}{\zeta_4}
\end{aligned}
\tag{B.113}
$$

4. A conversion (note: these overestimates may be appropriate for conversion back to valence quarks — although there is no guarantee that even their number density will be a falling function of x — or for the sum of sea + valence (which typically does fall with x), however note that no expression for $Q_\perp^2$ evolution is yet available for this type of PDF overestimate):

   (a) with $Q_\perp^2$: none.

   (b) with Q_A^2 and ζ_4 (similar B conversion with Q_B^2 and ζ_4):

$$
\begin{aligned}
\mathrm{d}\hat{\mathcal{A}}_{\text{conv A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} \, s_{AB} \, \hat{R}_f \, \mathrm{d}Q_A^2 \mathrm{d}\zeta_4 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \mathrm{d}\zeta_4 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \mathrm{d}\zeta_4
\end{aligned}
\tag{B.114}
$$

B.10 Zeta Integrals and Generation of Trial Zeta

The corresponding trial ζ integrals are:

$$
I_{\zeta_1} = \int_{\zeta_a}^{\zeta_b} \mathrm{d}\zeta_1 \frac{1}{\zeta_1(1-\zeta_1)} = \ln\left(\frac{\zeta_1}{1-\zeta_1}\right)\Bigg|_{\zeta_a}^{\zeta_b} = \ln\left(\frac{\zeta_b(1-\zeta_a)}{\zeta_a(1-\zeta_b)}\right) \,, \tag{B.115}
$$

$$
I_{\zeta_{2/3}} = \int_{\zeta_a}^{\zeta_b} \mathrm{d}\zeta_{2/3} \frac{1}{1+\zeta_{2/3}} = \ln(1+\zeta_{2/3})\big|_{\zeta_a}^{\zeta_b} = \ln\left(\frac{1+\zeta_b}{1+\zeta_a}\right) \,, \tag{B.116}
$$

$$
I_{\zeta_4^{-a}} = \int_{\zeta_a}^{\zeta_b} \mathrm{d}\zeta_4 \zeta_4^{-a} = \begin{cases} \zeta_a^{-1} - \zeta_b^{-1} & \text{for } a = 2 \\ \ln \frac{\zeta_b}{\zeta_a} & \text{for } a = 1 \\ \zeta_b - \zeta_a & \text{for } a = 0 \end{cases} \,, \tag{B.117}
$$

using a to parametrise the power of $1/\zeta_4$ that appears in the various trial intergals.

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\text{min}}, \zeta)}{I_\zeta(\zeta_{\text{min}}, \zeta_{\text{max}})} \ , \tag{B.118}$$

the solutions are

$$\zeta_1 = \left[1 + \frac{1-\zeta_{\text{min}}}{\zeta_{\text{min}}} \left(\frac{\zeta_{\text{min}}(1-\zeta_{\text{max}})}{\zeta_{\text{max}}(1-\zeta_{\text{min}})}\right)^{\mathcal{R}_{\zeta_1}}\right]^{-1} \ , \tag{B.119}$$

$$\zeta_{2/3} = (1+\zeta_{\text{min}}) \left(\frac{1+\zeta_{\text{max}}}{1+\zeta_{\text{min}}}\right)^{\mathcal{R}_{\zeta_{2/3}}} - 1 \ , \tag{B.120}$$

$$\zeta_4 = \begin{cases} \frac{\zeta_{\text{max}}\zeta_{\text{min}}}{\zeta_{\text{max}}+\mathcal{R}(\zeta_{\text{min}}-\zeta_{\text{max}})} & \text{for } a=2 \\ \zeta_{\text{min}} \left(\frac{\zeta_{\text{max}}}{\zeta_{\text{min}}}\right)^{\mathcal{R}} & \text{for } a=1 \\ \zeta_{\text{min}} + \mathcal{R}(\zeta_{\text{max}}-\zeta_{\text{min}}) & \text{for } a=0 \end{cases} \ . \tag{B.121}$$

B.11 Generation of Trial Evolution Scale

The integral over the evolution is scale is

$$\int_{Q^2_{E\,\text{new}}}^{Q^2_{E\,\text{F}}} \frac{\mathrm{d}Q^2_E}{Q^2_E} = \ln Q^2_E\Big|^{Q^2_{E\,\text{F}}}_{Q^2_{E\,\text{new}}} = \ln \frac{Q^2_{E\,\text{F}}}{Q^2_{E\,\text{new}}} \tag{B.122}$$

The next trial scale is found by solving the equation

$$\hat{\Delta}(Q^2_E, Q^2_{E\,\text{new}}) = \mathcal{R} \tag{B.123}$$

for $Q^2_{E\,\text{new}}$.

For constant trial $\hat{\alpha}_s$, the solutions are (using $\alpha = 1$ to denote the $xf \leq \text{const}$ overestimate forms and $\alpha = 0$ for the $f \leq \text{const}$ ones, and $\hat{R}_{x^\alpha f}$ to emphasize that the corresponding accept probability should be $R_{x^\alpha f}/\hat{R}_f$):

1. Soft with $Q^2_\perp$ and ζ_1:

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s \mathcal{C}}{2\pi} I_{\zeta_1} \hat{R}_f \ln\left(\frac{Q^2_\perp}{Q^2_{\perp\text{new}}}\right)\right) \ \Leftrightarrow \ Q^2_{\perp\text{new}} = Q^2_\perp \mathcal{R}^{\frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1}\hat{R}_f}} \tag{B.124}$$

2. A gluon collinear with $Q^2_\perp$ and ζ_3 (similar B gluon collinear with ζ_2):

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s \mathcal{C}}{2\pi} I_{\zeta_{2/3}} \hat{R}_{xf} \ln\left(\frac{Q^2_\perp}{Q^2_{\perp\text{new}}}\right)\right) \ \Leftrightarrow \ Q^2_{\perp\text{new}} = Q^2_\perp \mathcal{R}^{\frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_{2/3}}\hat{R}_{xf}}} \tag{B.125}$$

3. A gluon splitting

   (a) with $Q_\perp^2$ and ζ_3 (similar B gluon splitting with ζ_2):

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s\mathcal{C}}{4\pi} I_{\zeta_{2/3}} \hat{R}_f \ln\left(\frac{Q_\perp^2}{Q_{\perp\text{new}}^2}\right)\right) \Leftrightarrow Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s\mathcal{C}}\frac{1}{I_{\zeta_1}\hat{R}_f}} \tag{B.126}$$

   (b) with Q_A^2 and ζ_4 (similar B gluon splitting with Q_B^2 and ζ_4):

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s\mathcal{C}}{4\pi} I_{\zeta_4\,1} \hat{R}_{x^\alpha f} \ln\left(\frac{Q_A^2}{Q_{A\text{new}}^2}\right)\right) \Leftrightarrow Q_{A\text{new}}^2 = Q_A^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s\mathcal{C}}\frac{1}{I_{\zeta_4\,1}\hat{R}_{x^\alpha f}}} \tag{B.127}$$

4. A conversion

   (a) with $Q_\perp^2$ and ζ_3 (similar B conversion with ζ_2):

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s\mathcal{C}}{4\pi} I_{\zeta_{2/3}} \hat{R}_{xf} \ln\left(\frac{Q_\perp^2}{Q_{\perp\text{new}}^2}\right)\right) \Leftrightarrow Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s\mathcal{C}}\frac{1}{I_{\zeta_{2/3}}\hat{R}_{xf}}} \tag{B.128}$$

   (b) with Q_A^2 and ζ_4 (similar B conversion with Q_B^2 and ζ_4):

$$\mathcal{R} = \exp\left(-\frac{\hat{\alpha}_s\mathcal{C}}{4\pi} I_{\zeta_4\,2} \hat{R}_{x^\alpha f} \ln\left(\frac{Q_A^2}{Q_{A\text{new}}^2}\right)\right) \Leftrightarrow Q_{A\text{new}}^2 = Q_A^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s\mathcal{C}}\frac{1}{I_{\zeta_4\,2}\hat{R}_{x^\alpha f}}} \tag{B.129}$$

Running of the Coupling: We use

$$\alpha_s(Q_E^2) = \frac{1}{b_0 \ln\left(\frac{k_R^2 Q_E^2}{\Lambda^2}\right)} \tag{B.130}$$

to write the evolution kernels as

$$\mathrm{d}\hat{\mathcal{A}} = \text{Com}\,\frac{\mathrm{d}Q_E^2}{Q_E^2 \ln\left(\frac{k_R^2 Q_E^2}{\Lambda^2}\right)} \Leftrightarrow \hat{\mathcal{A}} = \text{Com}\,\ln\left(\frac{\ln\left(\frac{k_R^2 Q_E^2}{\Lambda^2}\right)}{\ln\left(\frac{k_R^2 Q_{E\,\text{new}}^2}{\Lambda^2}\right)}\right). \tag{B.131}$$

The solutions for the next trial scale $Q_{E\,\text{new}}^2$ are therefore

$$\mathcal{R} = \exp\left(-\text{Com}\,\ln\left(\frac{\ln\left(\frac{k_R^2 Q_E^2}{\Lambda^2}\right)}{\ln\left(\frac{k_R^2 Q_{E\,\text{new}}^2}{\Lambda^2}\right)}\right)\right) \Leftrightarrow Q_{E\,\text{new}}^2 = \frac{\Lambda^2}{k_R^2}\left(\frac{k_R^2 Q_E^2}{\Lambda^2}\right)^{\mathcal{R}^{1/\text{Com}}}. \tag{B.132}$$

To include two-loop running we use one-loop running as above and veto with

$$\frac{\alpha_s^{(2)}(Q_E^2, \Lambda_{\text{QCD}}^{(2)})}{\alpha_s^{(1)}(Q_E^2, \Lambda_{\text{QCD}}^{(2)})} \tag{B.133}$$

B.12 Inverse Transforms

The inversions are for $Q_\perp^2$ and ζ_1

$$s_{jb} = \frac{Q_\perp^2}{\zeta_1} \tag{B.134}$$

$$s_{aj} = \frac{Q_\perp^2 + \zeta_1 s_{AB}}{1 - \zeta_1} \tag{B.135}$$

and for $Q_\perp^2$ and ζ_2

$$s_{aj} = \zeta_2 s_{AB} \tag{B.136}$$

$$s_{jb} = \frac{Q_\perp^2(1 + \zeta_2)}{\zeta_2 - Q_\perp^2/s_{AB}} \tag{B.137}$$

and for $Q_\perp^2$ and ζ_3

$$s_{jb} = \zeta_3 s_{AB} \tag{B.138}$$

$$s_{aj} = \frac{Q_\perp^2(1 + \zeta_3)}{\zeta_3 - Q_\perp^2/s_{AB}} \,, \tag{B.139}$$

and for Q_A^2 and ζ_4

$$s_{aj} = Q_A^2 \tag{B.140}$$

$$s_{jb} = s_{AB}(\zeta_4 - 1) - Q_A^2 \,, \tag{B.141}$$

and for Q_B^2 and ζ_4

$$s_{jb} = Q_B^2 \tag{B.142}$$

$$s_{aj} = s_{AB}(\zeta_4 - 1) - Q_B^2 \,. \tag{B.143}$$

C Initial–Final Evolution Equations

C.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $AK \to ajk$, respectively. Conservation of energy and momentum implies $p_a + p_K = p_A + p_j + p_k$ or equivalently $p_a - p_j - p_k = p_A - p_K$. For general masses, with the notation $s_{ij} \equiv 2p_i \cdot p_j$, the relation between the pre-and post-branching invariants is thus

$$s_{AK} - m_A^2 - m_K^2 \;=\; s_{aj} + s_{ak} - s_{jk} - m_a^2 - m_j^2 - m_k^2 \,. \tag{C.1}$$

If all partons are massless, or if parton j is a gluon (gluon emission, such that $m_a = m_A$, $m_k = m_K$, and $m_j = 0$), the relation simplifies to

$$s_{AK} \;=\; s_{aj} + s_{ak} - s_{jk} \,. \tag{C.2}$$

For gluon K splitting to massive quarks j and k, the relation for arbitrary $m_a = m_A$ and $m_j = m_k$ is

$$s_{AK} \;=\; s_{aj} + s_{ak} - s_{jk} - 2m_j^2 \,. \tag{C.3}$$

For a gluon A backwards-evolving to a quark a (gluon conversion), we have (irrespective of the recoiler mass $m_K = m_k$),

$$s_{AK} \;=\; s_{aj} + s_{ak} - s_{jk} - m_a^2 - m_j^2 \,, \tag{C.4}$$

where we leave the possibility open of defining the new incoming quark to have a virtuality different from that of the (on-shell) final-state one. For a quark A backwards-evolving to a gluon a (quark conversion), we have (irrespective of the recoiler mass $m_K = m_k$),

$$s_{AK} - m_A^2 \;=\; s_{aj} + s_{ak} - s_{jk} - m_j^2 \,. \tag{C.5}$$

which for the nominal case of $m_A = m_j$ reduces to the formula for the fully massless case given above.

For dimensionless equivalents, we normalise by the largest invariant, $2p_a \cdot (p_j + p_k) = s_{aj} + s_{ak}$, hence

$$y_{aj} \;=\; \frac{s_{aj}}{s_{aj} + s_{ak}} \,, \tag{C.6}$$

$$y_{jk} \;=\; \frac{s_{jk}}{s_{aj} + s_{ak}} \,, \tag{C.7}$$

$$y_{ak} \;=\; \frac{s_{ak}}{s_{aj} + s_{ak}} \,, \tag{C.8}$$

where, for massless partons, the denominators can also be written $s_{aj} + s_{ak} = s_{AK} + s_{jk}$ and the dimensionless momentum-conservation relation implies

$$y_{AK} \;=\; 1 - y_{jk} \,, \tag{C.9}$$

$$y_{ak} \;=\; 1 - y_{aj} \,, \tag{C.10}$$

$$y_{aj} + y_{ak} \;=\; y_{AK} + y_{jk} \;=\; 1 \,. \tag{C.11}$$

C.2 Crossing Relations

Compared to the FF case, crossing symmetry for the case when recoiling parton i is crossed to be identified with a, implies

$$y_{ij} \quad \to \quad \frac{-s_{aj}}{-s_{AK}} \equiv \frac{y_{aj}}{y_{AK}} \tag{C.12}$$

$$y_{jk} \quad \to \quad \frac{s_{jk}}{-s_{AK}} \equiv \frac{-y_{jk}}{y_{AK}} \tag{C.13}$$

$$y_{ik} \quad \to \quad \frac{-s_{ak}}{-s_{AK}} \equiv \frac{y_{ak}}{y_{AK}}. \tag{C.14}$$

Crossings for which parton k is identifed with a and parton i becomes parton k (e.g., for the GQemitIF antenna function) have:

$$y_{ij} \quad \to \quad \frac{s_{jk}}{-s_{AK}} \equiv \frac{-y_{jk}}{y_{AK}} \tag{C.15}$$

$$y_{jk} \quad \to \quad \frac{-s_{aj}}{-s_{AK}} \equiv \frac{y_{aj}}{y_{AK}} \tag{C.16}$$

$$y_{ik} \quad \to \quad \frac{-s_{ak}}{-s_{AK}} \equiv \frac{y_{ak}}{y_{AK}}. \tag{C.17}$$

Note also that outgoing particles are mapped to incoming particles with the opposite helicities, so that, e.g., the IF antenna function for $(\bar{q}_A^+ \bar{q}_K^+ \to \bar{q}_a^+ g_j^+ \bar{q}_k^+)$ is obtained from the FF one for $(q_I^- \bar{q}_K^+ \to q_i^- g_j^+ \bar{q}_k^+)$.

Finally, as for the case of the II functions, note that the initial-state parts of VINCIA's antenna functions are 'sectorised' since there is no sector for 'emission into the initial state'. The initial-state functions with gluon parents are therefore mostly not simple crossings of the corresponding final-state ones. The IF case is a hybrid, with the initial-state legs being sectorised, while the final-state legs are global.

C.3 Antenna Functions

Note: the initial-state collinear limits can be examined by using $z = x_i = x_A/x_a \to (1 - y_{jk}) = y_{AK}$. The final-state collinear limits can be examined using $z = x_k = y_{ak}/y_{AK} \to 1 - y_{aj} = y_{ak}$.

C.3.1 QQemitIF

The helicity-averaged antenna function is:

$$\begin{aligned}
a(q_A q_K \to q_a g_j q_k) = &\frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^2 + (1-y_{jk})^2}{y_{aj} y_{jk}} \right. \\
&- \frac{2\mu_a^2}{y_{aj}^2} \left((1-y_{jk}) \left(1 - \frac{y_{aj}}{2} \right) - \frac{y_{aj}}{2} (1-y_{aj}) \right) \\
&- \frac{2\mu_k^2}{y_{jk}^2} \left(1 - \frac{y_{jk}}{4} (2-y_{jk}) \left(2 + \frac{y_{aj}^2}{1-y_{aj}} \right) \right) \\
&\left. + \frac{1}{2} (2-y_{aj})(2-y_{jk}) \right] . \qquad \text{(C.18)}
\end{aligned}$$

The individual helicity contributions are:

$$a(++ \to +++) = \frac{1}{s_{AK}} \left[\frac{1}{y_{aj} y_{jk}} - \frac{\mu_a^2}{y_{aj}^2} - \frac{\mu_k^2}{(1-y_{aj}) y_{jk}^2} \right] , \tag{C.19}$$

$$\begin{aligned} a(++ \to +-+) &= \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^2 + [(1-y_{jk})^2 - 1](1-y_{aj})^2}{y_{aj} y_{jk}} \right. \\ &\quad \left. - \frac{\mu_a^2 (1-y_{jk}-y_{aj})^2}{y_{aj}^2} - \frac{\mu_k^2 (1-y_{aj})(1-y_{jk})^2}{y_{jk}^2} \right] , \end{aligned} \tag{C.20}$$

$$a(++ \to --+) = \frac{1}{s_{AK}} \left[\frac{\mu_a^2 y_{jk}^2}{y_{aj}^2} \right] , \tag{C.21}$$

$$a(++ \to ++-) = \frac{1}{s_{AK}} \left[\frac{\mu_k^2 y_{aj}^2}{(1-y_{aj}) y_{jk}^2} \right] , \tag{C.22}$$

$$a(+- \to ++-) = \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^2}{y_{aj} y_{jk}} - \frac{\mu_a^2 (1-y_{aj})}{y_{aj}^2} - \frac{\mu_k^2 (1-y_{aj})}{y_{jk}^2} \right] , \tag{C.23}$$

$$a(+- \to +--) = \frac{1}{s_{AK}} \left[\frac{(1-y_{jk})^2}{y_{aj} y_{jk}} - \frac{\mu_a^2 (1-y_{jk})^2}{y_{aj}^2} - \frac{\mu_k^2 (1-y_{jk})^2}{y_{jk}^2 (1-y_{aj})} \right] , \tag{C.24}$$

$$a(+- \to ---) = \frac{1}{s_{AK}} \left[\frac{\mu_a^2 y_{jk}^2}{y_{aj}^2} \right] , \tag{C.25}$$

$$a(+- \to +-+) = \frac{1}{s_{AK}} \left[\frac{\mu_k^2 y_{aj}^2}{y_{jk}^2 (1-y_{aj})} \right] . \tag{C.26}$$

Note 1: the sum of the ++ antenna functions has the same singularities as the sum of the +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note 2: the direct crossing of the second (+-+) antenna has a zero across the diagonal of the IF phase space, corresponding to points with $s_{ak} = s_{jk}$ or equivalently $s_{AK} = s_{aj}$. The former condition is satisfied when $(p_a - p_j) \cdot p_k \to 0$. Not having come up with a good physical reason why that antenna function should go to zero there, we have chosen to add nonsingular terms as shown above.

Note 3: The $++ \to +-+$, $+- \to ++-$ and $+- \to +--$ antennae have all had non-singular pieces proportional to the mass terms added to guarantee positive definiteness over the full phase space for any choices of masses.

C.3.2 QGemitIF

The helicity-averaged antenna function is:

$$
\begin{aligned}
a(q_A g_K \to q_a g_j g_k) = & \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3+(1-y_{jk})^2}{y_{aj}y_{jk}} + (1-\alpha)\frac{1-2y_{aj}}{y_{jk}}\right. \\
& - \frac{2\mu_a^2}{y_{aj}^2}\left((1-y_{jk}) - \frac{y_{aj}}{2}\left[(1-y_{aj})-(2-y_{jk})^2\right]\right) \\
& \left. + \frac{3}{2} + y_{aj} - \frac{y_{jk}}{2} - \frac{y_{aj}^2}{2}\right]
\end{aligned} \tag{C.27}
$$

The individual helicity contributions are:

$$
a(++\to+++) = \frac{1}{s_{AK}}\left[\frac{1}{y_{aj}y_{jk}} + (1-\alpha)\frac{1-2y_{aj}}{y_{jk}} - \frac{\mu_a^2}{y_{aj}^2}\right], \tag{C.28}
$$

$$
a(++\to+-+) = \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3+(1-y_{jk})^2-1}{y_{aj}y_{jk}} - \frac{\mu_a^2(1-y_{jk}-y_{aj})^2(1-y_{aj})}{y_{aj}^2} + 3 - y_{aj}^2\right] \tag{C.29}
$$

$$
a(++\to--+) = \frac{1}{s_{AK}}\left[\frac{\mu_a^2 y_{jk}^2}{y_{aj}^2}\right] \tag{C.30}
$$

$$
a(+-\to++-) = \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3}{y_{aj}y_{jk}} - \frac{\mu_a^2(1-y_{aj})^2}{y_{aj}^2}\right], \tag{C.31}
$$

$$
a(+-\to+--) = \frac{1}{s_{AK}}\left[\frac{(1-y_{jk})^2}{y_{aj}y_{jk}} + (1-\alpha)\frac{1-2y_{aj}}{y_{jk}} - \frac{\mu_a^2(1-y_{jk})^2}{y_{aj}^2} + 2y_{aj} - y_{jk}\right]
$$

$$
a(+-\to---) = \frac{1}{s_{AK}}\left[\frac{\mu_a^2 y_{jk}^2}{y_{aj}^2}\right] \tag{C.32}
$$

Note that the sum of the ++ antenna functions has the same singularities as the sum of the +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note 3: the nonsingular terms are to ensure positive-definite functions which do not vanish at the arbitrary line across the diagonal of the phase space while still vanishing for hard-collinear helicity flips.

Note 4: The $++\to+-+$ and $+-\to++-$ antennae have had non-singular pieces proportional to the mass terms added to guarantee positive definiteness over the full phase space for any choices of masses.

C.3.3 GQemitIF

Note: the $1/(y_{aj}(1-y_{jk}))$ denominators in the "sector terms" on the initial-state side (necessary to ensure that the full initial-state gluon-collinear limits are reproduced since there is no global antenna function for emission into the initial state) could in principle equally well be $1/(y_{aj}(1-y_{jk}+y_{aj}))$ which would slightly dampen their numerical values outside the limit, without modifying the limit itself.

The helicity-averaged antenna function is:

$$\begin{aligned} a(g_A q_K \to g_a g_j q_k) = & \frac{1}{s_{AK}} \left[\frac{(1-y_{jk})^3 + (1-y_{aj})^2}{y_{aj} y_{jk}} + \frac{1+y_{jk}^3}{y_{aj}(1-y_{jk})} \right. \\ & - \frac{2\mu_k^2}{y_{jk}^2} \left(1 - \frac{y_{jk}}{4}(3 - 3y_{jk}^2 + y_{jk}^3) \left(2 + \frac{y_{aj}^2}{1-y_{aj}} \right) \right) \\ & \left. \frac{1}{2}(2-y_{aj})(3-y_{jk}+y_{jk}^2) \right] . \end{aligned} \tag{C.33}$$

Since the gluon is in the initial state, the individual helicity contributions are crossings of corresponding sector FF functions, up to non-singular terms.

$$a(++ \to +++) = \frac{1}{s_{AK}}\left[\frac{1}{y_{aj}y_{jk}} + \frac{1}{y_{aj}(1-y_{jk})} - \frac{\mu_k^2}{y_{jk}^2(1-y_{aj})}\right], \tag{C.34}$$

$$a(++ \to +-+) = \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^2 + [(1-y_{jk})^3 - 1](1-y_{aj})^2}{y_{aj}y_{jk}} - \frac{\mu_k^2(1-y_{aj})(1-y_{jk})^3}{y_{jk}^2}\right], \tag{C.35}$$

$$a(++ \to --+) = \frac{1}{s_{AK}}\frac{y_{jk}^3}{y_{aj}(1-y_{jk})}, \tag{C.36}$$

$$a(++ \to ++-) = \frac{1}{s_{AK}}\left[\frac{\mu_k^2 y_{aj}^2}{y_{jk}^2(1-y_{aj})}\right], \tag{C.37}$$

$$a(+- \to ++-) = \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^2}{y_{aj}y_{jk}} + \frac{1}{y_{aj}(1-y_{jk})} - \frac{\mu_k^2(1-y_{aj})}{y_{jk}^2}\right], \tag{C.38}$$

$$a(+- \to +--) = \frac{1}{s_{AK}}\left[\frac{(1-y_{jk})^3}{y_{aj}y_{jk}} - \frac{\mu_k^2(1-y_{jk})^3}{y_{jk}^2(1-y_{aj})}\right], \tag{C.39}$$

$$a(+- \to ---) = a(++ \to --+), \tag{C.40}$$

$$a(+- \to +-+) = a(++ \to ++-). \tag{C.41}$$

Note: the nonsingular terms for the helicity-flip antenna are chosen such that the function still goes to zero in the hard-collinear limits but allows it to be non-zero in the hard part of phase space.

C.3.4 GGemitIF

The GGemitIF functions are essentially hybrids between sector antenna functions for the initial-state singularities and global ones for the final-state legs.

Note: similarly to the GQemitIF functions, the $1/(y_{aj}(1-y_{jk}))$ denominators in the "sector terms" on the initial-state side (necessary to ensure that the full initial-state gluon-collinear limits are reproduced since there is no global antenna function for emission into the initial state) could in principle equally well be $1/(y_{aj}(1-y_{jk}+y_{aj}))$ which would slightly dampen their numerical values outside the limit, without modifying the limit itself.

The helicity-averaged antenna function is:

$$a(g_A g_K \to g_a g_j q_k) \;=\; \tag{C.42}$$
$$\frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3+(1-y_{jk})^3}{y_{aj}y_{jk}}+\frac{1+y_{jk}^3}{y_{aj}(1-y_{jk})}+(1-\alpha)\frac{1-2y_{aj}}{y_{jk}}+3-2y_{jk}\right]\,.$$

The helicity contributions are:

$$a(++\to+++) \;=\; \frac{1}{s_{AK}}\left[\frac{1}{y_{aj}y_{jk}}+(1-\alpha)\frac{1-2y_{aj}}{y_{jk}}+\frac{1}{y_{aj}(1-y_{jk})}\right]\,, \tag{C.43}$$

$$a(++\to+-+) \;=\; \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3+(1-y_{jk})^3-1}{y_{aj}y_{jk}}+6-3y_{aj}-3y_{jk}+y_{aj}y_{jk}\right], \tag{C.44}$$

$$a(+-\to++-) \;=\; \frac{1}{s_{AK}}\left[\frac{(1-y_{aj})^3}{y_{aj}y_{jk}}+\frac{1}{y_{aj}(1-y_{jk})}\right]\,, \tag{C.45}$$

$$a(+-\to+--) \;=\; \frac{1}{s_{AK}}\left[\frac{(1-y_{jk})^3}{y_{aj}y_{jk}}+(1-\alpha)\frac{1-2y_{aj}}{y_{jk}}+3y_{aj}-y_{jk}-y_{aj}y_{jk}\right]. \tag{C.46}$$

Note: the nonsingular terms for the helicity-flip antenna are chosen such that it remains positive-definite over the phase space with $p_\perp^2 < s_{AK}$ and still goes to zero in the hard-collinear limits.

Note 2: the last function includes some quadratic terms to remain positive definite.

The two additional antennae, with helicity flips on the incoming gluon leg (i.e., with parton j inheriting the a helicity, rather than A) are the same as for the GQemitIF case:

$$a(++\to--+) \;=\; \frac{1}{s_{AK}}\frac{y_{jk}^3}{y_{aj}(1-y_{jk})}\,, \tag{C.47}$$

$$a(+-\to---) \;=\; a(++\to--+)\,. \tag{C.48}$$

C.3.5 XGsplitIF

The XGsplitIF functions are essentially identical to the final-state gluon-splitting antennae, the only difference being that the recoiler is now an initial-state parton. The helicity average (for unpolarised partons, including an optional correction term for splitting to massive quarks) is:

$$a(X_A g_K \to X_a \bar{q}_j q_k) = \frac{1}{2m_{jk}^2}\left[y_{ak}^2+y_{aj}^2+\frac{2m_j^2}{m_{jk}^2}\right]\,. \tag{C.49}$$

The helicity contributions are:

$$a(X+ \to X-+) = \frac{1}{2m_{jk}^2}\left[y_{ak}^2 - \frac{m_j^2 y_{ak}}{m_{jk}^2(1-y_{ak})}\right] , \quad (C.50)$$

$$a(X+ \to X+-) = \frac{1}{2m_{jk}^2}\left[y_{aj}^2 - \frac{m_j^2 y_{aj}}{m_{jk}^2(1-y_{aj})}\right] , \quad (C.51)$$

$$a(X+ \to X++) = \frac{m_j^2}{2m_{jk}^4}\left[\frac{y_{aj}}{(1-y_{aj})} + \frac{y_{ak}}{(1-y_{ak})} + 2\right] . \quad (C.52)$$

Note, in principle the exact crossing corresponds to performing the substitution $m_{jk}^2 \to m_{jk}^2 y_{AK}^2$ in the denominators above, which is neglected since the $y_{jk} \to 0$ limit corresponds to $y_{AK} \to 1$.

C.3.6 QXsplitIF

The QXsplitIF functions are essentially identical to the QXsplitII functions, the only difference being that the recoiler is now a final-state parton.

The helicity-averaged antenna function is:

$$a(q_A X_K \to g_a \bar{q}_j X_k) = \frac{1}{s_{AK}}\left[\frac{y_{AK}^2 + (1-y_{AK})^2}{y_{aj}} + \frac{2\mu_j^2 y_{AK}}{y_{aj}^2}\right] . \quad (C.53)$$

The individual helicity contributions are:

$$a(+X \to +-X) = \frac{1}{s_{AK}}\left[\frac{y_{AK}^2}{y_{aj}} - \frac{\mu_j^2 y_{AK}^2}{y_{aj}^2(1-y_{AK})}\right] , \quad (C.54)$$

$$a(+X \to --X) = \frac{1}{s_{AK}}\left[\frac{(1-y_{AK})^2}{y_{aj}} - \frac{\mu_j^2(1-y_{AK})}{y_{aj}^2}\right] , \quad (C.55)$$

$$a(+X \to ++X) = \frac{1}{s_{AK}}\left[\frac{\mu_j^2}{y_{aj}^2(1-y_{AK})}\right] , \quad (C.56)$$

$$a(-X \to -+X) = a(+X \to +-X) , \quad (C.57)$$

$$a(-X \to ++X) = a(+X \to --X) , \quad (C.58)$$

$$a(-X \to --X) = a(+X \to ++X) . \quad (C.59)$$

C.3.7 GXconvIF

The GXconvIF functions are essentially identical to the GXconvII functions, the only difference being that the recoiler is now a final-state parton.

The helicity-averaged antenna function is:

$$a(g_A X_K \to q_a q_j X_k) \;=\; \frac{1}{2s_{AK}}\left[\frac{1+(1-y_{AK})^2}{y_{AK}(y_{aj}-2\mu_j^2)} - \frac{2\mu_j^2 y_{AK}}{(y_{aj}-2\mu_j^2)^2}\right] . \tag{C.60}$$

The individual helicity contributions are:

$$a(+X \to ++X) \;=\; \frac{1}{2s_{AK}}\left[\frac{1}{y_{AK}(y_{aj}-2\mu_j^2)} - \frac{\mu_j^2 y_{AK}}{(y_{aj}-2\mu_j^2)^2(1-y_{AK})}\right] , \tag{C.61}$$

$$a(+X \to --X) \;=\; \frac{1}{2s_{AK}}\left[\frac{(1-y_{AK})^2}{y_{AK}(y_{aj}-2\mu_j^2)} - \frac{\mu_j^2 y_{AK}(1-y_{AK})}{(y_{aj}-2\mu_j)^2}\right] , \tag{C.62}$$

$$a(+X \to +-X) \;=\; \frac{1}{2s_{AK}}\left[\frac{\mu_j^2}{(y_{aj}-2\mu_j^2)^2}\frac{y_{AK}^3}{1-y_{AK}}\right] , \tag{C.63}$$

$$a(-X \to --X) \;=\; a(+X \to ++X) , \tag{C.64}$$

$$a(-X \to ++X) \;=\; a(+X \to --X) , \tag{C.65}$$

$$a(-X \to -+X) \;=\; a(+X \to +-X) . \tag{C.66}$$

C.4 Evolution Variables

The evolution variables we use are

$$Q_\perp^2 = \frac{s_{aj}s_{jk}}{s_{AK}+s_{jk}} , \tag{C.67}$$

$$Q_A^2 = s_{aj} , \tag{C.68}$$

$$Q_K^2 = s_{jk} . \tag{C.69}$$

Phase Space Boundaries: With $s_{AK} + s_{jk} = s_{aj} + s_{ak}$ and $s_{jk} = -s_{AK} + s_{aK} = (x_a - x_A)\sqrt{s}p_{K-}$, the phase space boundaries are $0 \le s_{jk} \le \frac{1-x_A}{x_A}s_{AK}$ and $0 \le s_{aj} \le s_{AK} + s_{jk}$. The maxima of the evolution variables are

$$Q^2_{\perp\,\text{max}} = \frac{1-x_A}{x_A}s_{AK} , \tag{C.70}$$

$$Q^2_{A\,\text{max}} = s_{AK}/x_A , \tag{C.71}$$

$$Q^2_{K\,\text{max}} = \frac{1-x_A}{x_A}s_{AK} . \tag{C.72}$$

C.5 Zeta Definitions

The choices are

$$\zeta_1 = \frac{s_{jk} + s_{AK}}{s_{AK}} = \frac{x_a}{x_A} \;, \tag{C.73}$$

$$\zeta_2 = \frac{s_{aj}}{s_{AK} + s_{jk}} \;, \tag{C.74}$$

$$\tag{C.75}$$

To accomodate for the different phase space limits of the global and local maps, the phase space boundaries of ζ are defined in terms of s_{jk+}, which is given by

$$s_{jk+} = \begin{cases} \left(\frac{x_{a+}}{x_A} - 1\right) s_{AK} & \text{Local map} \\ \left(\frac{x_{a+}}{x_A}\frac{s_{AB}}{s_{AB}-s_{BK}} - 1)\right) s_{AK} & \text{Global map} \end{cases} \tag{C.76}$$

where x_{a+} is the remaining available partonic momentum fraction. The integration boundaries for the ζ variables are

$$\zeta_{1-}(Q_\perp^2) = \frac{Q_\perp^2 + s_{AK}}{s_{AK}} \qquad \zeta_{1+}(Q_\perp^2) = \frac{s_{AK} + s_{jk+}}{s_{AK}} \;, \tag{C.77}$$

$$\zeta_{2-}(Q_\perp^2) = \frac{Q_\perp^2}{s_{jk+}} \qquad \zeta_{2+}(Q_\perp^2) = 1 \;, \tag{C.78}$$

$$\zeta_{1-}(Q_A^2) = \max\left(1, \frac{Q_A^2}{s_{AK}}\right) \qquad \zeta_{1+}(Q_A^2) = \frac{s_{AK} + s_{jk+}}{s_{AK}} \;, \tag{C.79}$$

$$\zeta_{2-}(Q_K^2) = 0 \qquad \zeta_{2+}(Q_K^2) = 1 \;. \tag{C.80}$$

C.6 Jacobians

The Jacobians for the transformation from the phase-space variables, (s_{aj}, s_{jk}), to the shower variables, (Q_E, ζ), are

$$|J(Q_\perp^2, \zeta_1)| = \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \;, \tag{C.81}$$

$$|J(Q_\perp^2, \zeta_2)| = \frac{(s_{AK} + s_{jk})^2}{s_{aj}} \;, \tag{C.82}$$

$$|J(Q_A^2, \zeta_1)| = s_{AK} \;, \tag{C.83}$$

$$|J(Q_K^2, \zeta_2)| = s_{AK} + s_{jk} \;. \tag{C.84}$$

C.7 Trial Functions

The following trial functions are used (written to emphasise the overall factor $(s_{AK}+sjk)/s_{AK} = x_a/x_A$):

$$a_{\text{soft}} = \frac{1}{s_{AK}}\frac{2}{y_{aj}y_{jk}} \quad \Rightarrow \quad \hat{a}_{\text{soft}} = \frac{2(s_{AK}+s_{jk})}{s_{aj}s_{jk}}\frac{(s_{AK}+s_{jk})}{s_{AK}}\,, \tag{C.85}$$

$$a_{\text{g coll A}} = \frac{2}{s_{AK}}\frac{1}{y_{aj}(1-y_{jk})} \quad \Rightarrow \quad \hat{a}_{\text{g coll A}} = \frac{2(s_{AK}+s_{jk})}{s_{aj}s_{AK}}\frac{(s_{AK}+s_{jk})}{s_{AK}}\,, \tag{C.86}$$

$$a_{\text{conv A}} = \frac{1}{2}\frac{1}{s_{aj}}\frac{s_{ak}^2+s_{jk}^2}{s_{AK}^2} \quad \Rightarrow \quad \hat{a}_{\text{conv A}} = \frac{(s_{AK}+s_{jk})}{s_{aj}s_{AK}}\frac{(s_{AK}+s_{jk})}{s_{AK}}\,, \tag{C.87}$$

$$a_{\text{split A}} = \frac{1}{s_{AK}}\frac{s_{ak}}{s_{aj}} - \frac{2}{s_{AK}}\frac{s_{jk}}{s_{aj}}\frac{s_{AK}-s_{aj}}{s_{AK}+s_{jk}} \quad \Rightarrow \quad \hat{a}_{\text{split A}} = \frac{2}{s_{aj}}\frac{(s_{AK}+s_{jk})}{s_{AK}}\,, \tag{C.88}$$

$$a_{\text{split K}} = \frac{1}{2}\frac{1}{s_{jk}}\frac{s_{ak}^2+s_{aj}^2}{s_{AK}^2} \quad \Rightarrow \quad \hat{a}_{\text{split K}} = \frac{(s_{AK}+s_{jk})}{2s_{jk}s_{AK}}\frac{(s_{AK}+s_{jk})}{s_{AK}}\,. \tag{C.89}$$

C.8 Integration Kernels

The overestimate of the evolution integral is, for $(x^\alpha f) \leq \text{const}$ PDF overestimates

$$\hat{A}(Q^2_{E\,\text{F}}, Q^2_{E\,\text{new}}) = \int_{Q^2_{E\,\text{new}}}^{Q^2_{E\,\text{F}}} \frac{\hat{\alpha}_s \mathcal{C}}{4\pi}\, \frac{s_{AK}}{(s_{AK}+s_{jk})^2}\,\hat{a}\, \left(\frac{s_{AK}}{s_{AK}+s_{jk}}\right)^\alpha \hat{R}_f\, |J|\, \mathrm{d}Q_E^2 \mathrm{d}\zeta\, \frac{\mathrm{d}\phi}{2\pi}\,. \tag{C.90}$$

The integration kernels for the Q_E^2 integration are

1. Soft with $Q_\perp^2$ and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{soft}}(Q_\perp^2) = \frac{\hat{\alpha}_s\mathcal{C}}{2\pi}\hat{R}_f\frac{\mathrm{d}Q_\perp^2}{Q_\perp^2}\frac{\mathrm{d}\zeta_1}{\zeta_1^\alpha(\zeta_1-1)} \tag{C.91}$$

2. A gluon collinear with $Q_\perp^2$ and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{g coll A}}(Q_\perp^2) = \frac{\hat{\alpha}_s\mathcal{C}}{2\pi}\hat{R}_f\frac{\mathrm{d}Q_\perp^2}{Q_\perp^2}\frac{\mathrm{d}\zeta_1}{\zeta_1^\alpha} \tag{C.92}$$

3. A conversion

   (a) with $Q_\perp^2$ and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{conv A}}(Q_\perp^2) = \frac{\hat{\alpha}_s\mathcal{C}}{4\pi}\hat{R}_f\frac{\mathrm{d}Q_\perp^2}{Q_\perp^2}\frac{\mathrm{d}\zeta_1}{\zeta_1^\alpha} \tag{C.93}$$

(b) with Q_A^2 and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{conv A}}(Q_A^2) = \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \frac{\mathrm{d}\zeta_1}{\zeta_1^{\alpha}} \tag{C.94}$$

4. A gluon splitting

(a) with $Q_\perp^2$ and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{split A}}(Q_\perp^2) = \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \frac{\mathrm{d}\zeta_1}{\zeta_1^{1+\alpha}} \tag{C.95}$$

(b) with Q_A^2 and ζ_1:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{split A}}(Q_A^2) = \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{\mathrm{d}Q_A^2}{Q_A^2} \frac{\mathrm{d}\zeta_1}{\zeta_1^{1+\alpha}} \tag{C.96}$$

5. K gluon splitting

(a) with $Q_\perp^2$ and ζ_2, with $\alpha = 1$:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{split K}}(Q_\perp^2) = \frac{\hat{\alpha}_s \mathcal{C}}{8\pi} \hat{R}_f \frac{\mathrm{d}Q_\perp^2}{Q_\perp^2} \mathrm{d}\zeta_2 \tag{C.97}$$

(b) with Q_K^2 and ζ_2, with $\alpha = 1$:

$$\mathrm{d}\hat{\mathcal{A}}_{\text{split K}}(Q_K^2) = \frac{\hat{\alpha}_s \mathcal{C}}{8\pi} \hat{R}_f \frac{\mathrm{d}Q_K^2}{Q_K^2} \mathrm{d}\zeta_2 \tag{C.98}$$

C.9 Zeta Integrals and Generation of Trial Zeta

We have five different ζ integrals to solve,

$$I_{\zeta,\,\mathrm{quad}} = \int_{\zeta_a}^{\zeta_b} \zeta \mathrm{d}\zeta = \frac{\zeta_b^2 - \zeta_a^2}{2} \;, \tag{C.99}$$

$$I_{\zeta,\,\mathrm{lin}} = \int_{\zeta_a}^{\zeta_b} \mathrm{d}\zeta = \zeta_b - \zeta_a \;, \tag{C.100}$$

$$I_{\zeta,\,\mathrm{log}} = \int_{\zeta_a}^{\zeta_b} \frac{\mathrm{d}\zeta}{\zeta} = \ln \frac{\zeta_b}{\zeta_a} \;, \tag{C.101}$$

$$I_{\zeta,\,\mathrm{inv}} = \int_{\zeta_a}^{\zeta_b} \frac{\mathrm{d}\zeta}{\zeta^2} = \left. \frac{-1}{\zeta} \right|_{\zeta_a}^{\zeta_b} = \zeta_a^{-1} - \zeta_b^{-1} \;, \tag{C.102}$$

$$I_{\zeta,\,\mathrm{log2}} = \int_{\zeta_a}^{\zeta_b} \frac{\mathrm{d}\zeta}{\zeta - 1} = \ln \frac{\zeta_b - 1}{\zeta_a - 1} \;, \tag{C.103}$$

$$I_{\zeta,\,\mathrm{log3}} = \int_{\zeta_a}^{\zeta_b} \frac{\mathrm{d}\zeta}{\zeta(\zeta - 1)} = \ln \frac{(\zeta_b - 1)\zeta_a}{(\zeta_a - 1)\zeta_b} \;, \tag{C.104}$$

$$\tag{C.105}$$

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\mathrm{min}}, \zeta)}{I_\zeta(\zeta_{\mathrm{min}}, \zeta_{\mathrm{max}})} \;, \tag{C.106}$$

the solutions are

$$\zeta_{\mathrm{quad}} = \sqrt{\mathcal{R}_\zeta(\zeta_{\mathrm{min}}^2 - \zeta_{\mathrm{max}}^2) + \zeta_{\mathrm{max}}^2} \;, \tag{C.107}$$

$$\zeta_{\mathrm{lin}} = \mathcal{R}_\zeta(\zeta_{\mathrm{min}} - \zeta_{\mathrm{max}}) + \zeta_{\mathrm{max}} \;, \tag{C.108}$$

$$\zeta_{\mathrm{log}} = \zeta_{\mathrm{max}} \left(\frac{\zeta_{\mathrm{min}}}{\zeta_{\mathrm{max}}} \right)^{\mathcal{R}_\zeta} \;, \tag{C.109}$$

$$\zeta_{\mathrm{inv}} = \left[\mathcal{R}_\zeta(\zeta_{\mathrm{min}}^{-1} - \zeta_{\mathrm{max}}^{-1}) + \zeta_{\mathrm{max}}^{-1} \right]^{-1} \;, \tag{C.110}$$

$$\zeta_{\mathrm{log2}} = 1 + (\zeta_{\mathrm{min}} - 1) \left(\frac{\zeta_{\mathrm{max}} - 1}{\zeta_{\mathrm{min}} - 1} \right)^{\mathcal{R}} \;, \tag{C.111}$$

$$\zeta_{\mathrm{log3}} = \zeta_{\mathrm{min}} \left[\zeta_{\mathrm{min}} - (\zeta_{\mathrm{min}} - 1) \left(\frac{\zeta_{\mathrm{min}}}{\zeta_{\mathrm{min}} - 1} \frac{\zeta_{\mathrm{max}} - 1}{\zeta_{\mathrm{max}}} \right)^{\mathcal{R}} \right]^{-1} \;. \tag{C.112}$$

C.10 Generation of Trial Evolution Scale

The integral over the evolution is scale is

$$\int_{Q^2_{E\,\mathrm{new}}}^{Q^2_{E\,\mathrm{F}}} \frac{\mathrm{d}Q^2_E}{Q^2_E} = \ln Q^2_E \big|_{Q^2_{E\,\mathrm{new}}}^{Q^2_{E\,\mathrm{F}}} = \ln \frac{Q^2_{E\,\mathrm{F}}}{Q^2_{E\,\mathrm{new}}} \,. \tag{C.113}$$

The next trial scale is found by solving the equation

$$\hat{\Delta}(Q^2_E, Q^2_{E\,\mathrm{new}}) = \mathcal{R} \tag{C.114}$$

for $Q^2_{E\,\mathrm{new}}$. For constant trial $\hat{\alpha}_s$, the solutions are

1. Soft with $Q^2_\perp$ and ζ_1, for $\alpha = 1$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{2\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\log 3} \hat{R}_f}} \tag{C.115}$$

2. Soft with $Q^2_\perp$ and ζ_1, for $\alpha = 0$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{2\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\log 2} \hat{R}_f}} \tag{C.116}$$

3. A gluon collinear with $Q^2_\perp$ and ζ_1, for $\alpha = 1$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{2\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\log} \hat{R}_f}} \tag{C.117}$$

4. A gluon collinear with $Q^2_\perp$ and ζ_1, for $\alpha = 0$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{2\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\mathrm{lin}} \hat{R}_f}} \tag{C.118}$$

5. A conversion

   (a) with $Q^2_\perp$ and ζ_1, for $\alpha = 1$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{4\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\log} \hat{R}_f}} \tag{C.119}$$

   (b) with $Q^2_\perp$ and ζ_1, for $\alpha = 0$:

$$Q^2_{\perp\mathrm{new}} = Q^2_\perp \mathcal{R}^{\dfrac{4\pi}{\hat{\alpha}_s \mathcal{C}} \dfrac{1}{I_{\zeta,\mathrm{lin}} \hat{R}_f}} \tag{C.120}$$

(c) Same expressions apply for Q_A^2 and ζ_1.

6. A gluon splitting

   (a) with $Q_\perp^2$ and ζ_1:

$$Q^2_{\perp\text{new}} = Q_\perp^2 \mathcal{R}^{\frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1}(\alpha)\hat{R}_f}} \tag{C.121}$$

   (b) Same expression with Q_A^2 and ζ_1.

7. K gluon splitting

   (a) with $Q_\perp^2$ and ζ_2 with $\alpha = 1$:

$$Q^2_{\perp\text{new}} = Q_\perp^2 \mathcal{R}^{\frac{8\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_3,\,\text{lin}}\hat{R}_f}} \tag{C.122}$$

   (b) Same expression with Q_K^2 and ζ_2 with $\alpha = 1$.

Running of the Coupling: See initial-initial.

C.11 Inverse Transforms

The inversions are for $Q_\perp^2$ and ζ_1

$$s_{jk} = s_{AK}(\zeta_1 - 1) \tag{C.123}$$

$$s_{aj} = \frac{Q_\perp^2 \zeta_1}{\zeta_1 - 1} \tag{C.124}$$

and for Q_A^2 and ζ_1

$$s_{aj} = \frac{Q_A^2}{N_{s\text{I}}} \tag{C.125}$$

$$s_{jk} = (\zeta_1 - 1)s_{AK} \tag{C.126}$$

and for $Q_\perp^2$ and ζ_2

$$s_{jk} = \frac{Q_\perp^2}{\zeta_2} \tag{C.127}$$

$$s_{aj} = s_{AK}\zeta_2 + Q_\perp^2 \tag{C.128}$$

and for Q_K^2 and ζ_2

$$s_{jk} = Q_K^2 \tag{C.129}$$

$$s_{aj} = \zeta_2 \left(s_{AK} + Q_K^2\right) \tag{C.130}$$

D Accept Probabilities

D.1 Helicity Selection

For non maximally helicity violating (MHV) processes see [3]. MHV helicity selection can be simplified by studying the structure of MHV amplitudes, which are discussed in Section D.7 below. MHV amplitudes all have the following form:

$$|M_n^{FC}|_h^2 = |A_0(1^h, \ldots, n^h)|^2 \left| \sum_\sigma \frac{1}{\langle \sigma(1)\sigma(2) \rangle \ldots \langle \sigma(n)\sigma(1) \rangle} \text{CF}(\sigma(1)\ldots\sigma(n)) \right|^2 , \tag{D.1}$$

where $|M_n^{FC}|_h^2$ refers to the full colour (FC) MHV squared amplitude with helicity configuration label h, $A_0(1^h, \ldots, n^h)$ is some function of the helicities, σ is the relavent set of permutations, and CF is the colour factor. To polarise the hard process we want to calculate if:

$$\frac{\sum_{i=1}^h |M_n^{FC}|_i^2}{\sum_{h'} |M_n^{FC}|_{h'}^2} \geq R , \tag{D.2}$$

for h the helicity-configuration we are currently checking, and the sum over h' is a sum over all helicity configurations, which can be expanded as:

$$\sum_{h'} |M_n^{FC}|_{h'}^2 = \left(\sum_{h'} |A_0(1^{h'}, \ldots, n^{h'})|^2 \right) \left| \sum_\sigma \frac{1}{\langle \sigma(1)\sigma(2) \rangle \ldots \langle \sigma(n)\sigma(1) \rangle} \text{CF}(\sigma(1)\ldots\sigma(n)) \right|^2 . \tag{D.3}$$

Labelling the second term as $F(\sigma)$, we notice that equation (D.2) now reads:

$$\frac{\sum_{i=1}^h |M_n^{FC}|_i^2}{\sum_{h'} |M_n^{FC}|_{h'}^2} = \frac{\sum_{i=1}^h |A_0(1^i, \ldots, n^i)|^2}{\sum_{h'} |A_0(1^{h'}, \ldots, n^{h'})|^2} \frac{F(\sigma)}{F(\sigma)} = \frac{\sum_{i=1}^h |A_0(1^i, \ldots, n^i)|^2}{\sum_{h'} |A_0(1^{h'}, \ldots, n^{h'})|^2} \geq R , \tag{D.4}$$

and we can therefore use the much simpler expressions $|A_0(1^h, \ldots, n^h)|^2 = |M_n^{LC}|_h^2$ to polarise the process. That is, since the colour information is the same for each MHV helicity configuration we can factorise it out from the sum of matrix elements. QCD processes are non-chiral, so we explicitly calculate only half of the factors $|M_n^{LC}|_h^2$, since the other half are equal by parity.

We cannot do the above for 4-quark MHV amplitudes, because there is a second colour-connection when the two quarks have the same helicity. Hence the colour-factor depends on the helicity and cannot be factorised.

D.2 Smooth-Ordering Factor: P_{imp}

Note: this section is largely adapted from the discussion in [4].

In smooth ordering, the only quantity which must still be strictly ordered are the antenna invariant masses, a condition which follows from the nested antenna phase spaces and momentum conservation. Within each individual antenna, and between competing ones, the measure of

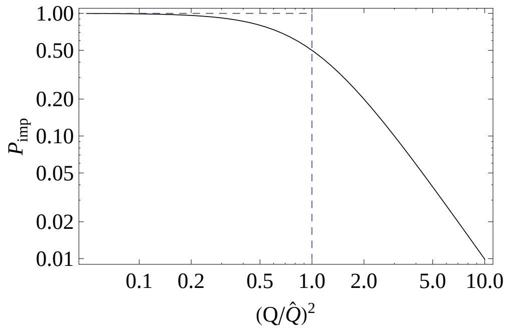

Figure 3: The smooth-ordering factor (*solid*) compared to a strong-ordering Θ function (*dashed*).

evolution time is still provided by the ordering variable, which we therefore typically refer to as the "evolution variable" in this context (rather than the "ordering variable"), in order to prevent confusion with the strong-ordering case. The evolution variable can in principle still be chosen to be any of the possibilities given above, though we shall typically use $2p_\perp$ for gluon emission and $m_{q\bar{q}}$ for gluon splitting.

In terms of an arbitrary evolution variable, Q, the smooth-ordering factor is [3]

$$P_{\rm imp}\left(Q^2,\hat{Q}^2\right) = \frac{\hat{Q}^2}{\hat{Q}^2+Q^2}~, \tag{D.5}$$

where Q is the evolution scale associated with the current branching, and $\hat{Q}$ measures the scale of the parton configuration before branching. A comparison to the strong-ordering step function is given in fig. 3, on a log-log scale. Since this factor is bounded by $0 \leq P_{\rm imp} \leq 1$, it can be applied as a simple accept/reject on each trial branching.

When switched on, smooth ordering is technically achieved as follows. After each accepted branching, the daughter antennae involved in that particular branching are allowed to restart their evolution from a scale nominally equivalent to their respective kinematic maximum. Trial branchings are then generated in the "unordered" part of phase space first, for those antennae only, while all other antennae in the event are "on hold", waiting for the scale to drop back down to normal ordering before the global event evolution is continued. The $P_{\rm imp}$ factor is applied as an extra multiplicative modification to the accept probability for each trial branching, in both the ordered and unordered regions of phase space.

In the strongly-ordered region of phase-space, $Q \ll \hat{Q}$, we may rewrite the $P_{\rm imp}$ factor as

$$P_{\rm imp} = \frac{1}{1+\frac{Q^2}{\hat{Q}^2}} \stackrel{Q<\hat{Q}}{=} 1 - \frac{Q^2}{\hat{Q}^2} + \ldots~. \tag{D.6}$$

Applying this to the $2 \to 3$ antenna function whose leading singularity is proportional to $1/Q^2$, we see that the overall correction arising from the $Q^2/\hat{Q}^2$ and higher terms is finite and of order $1/\hat{Q}^2$; a power correction. The LL singular behaviour is therefore not affected. However, at the multiple-emission level, the $1/\hat{Q}^2$ terms do modify the *subleading* logarithmic structure, starting from $\mathcal{O}(\alpha_s^2)$, as we shall return to below.

In the *unordered* region of phase-space, $Q > \hat{Q}$, we rewrite the P_{imp} factor as

$$P_{\mathrm{imp}} = \frac{\hat{Q}^2}{Q^2}\frac{1}{1+\frac{\hat{Q}^2}{Q^2}} \overset{Q\geq\hat{Q}}{=} \frac{\hat{Q}^2}{Q^2}\left(1-\frac{\hat{Q}^2}{Q^2}+\ldots\right) , \tag{D.7}$$

which thus effectively modifies the leading singularity of the LL $2 \to 3$ function from $1/Q^2$ to $1/Q^4$, removing it from the LL counting. The only effective terms $\propto 1/Q^2$ arise from finite terms in the radiation functions, if any such are present, multiplied by the P_{imp} factor. Only a matching to the full tree-level $2 \to 4$ functions would enable a precise control over these terms. Up to any given fixed order, this can effectively be achieved by matching to tree-level matrix elements. Matching beyond the fixed-order level is beyond the scope of the current treatment. We thus restrict ourselves to the observation that, at the LL level, smooth ordering is equivalent to strong ordering, with differences only appearing at the subleading level.

The effective $2 \to 4$ probability in the shower arises from a sum over two different $2 \to 3 \otimes 2 \to 3$ histories, each of which has the tree-level form

$$\hat{A}\, P_{\mathrm{imp}}\, A \;\propto\; \frac{1}{\hat{Q}^2}\frac{\hat{Q}^2}{\hat{Q}^2+Q^2}\frac{1}{Q^2} \;=\; \frac{1}{\hat{Q}^2+Q^2}\frac{1}{Q^2} , \tag{D.8}$$

thus we may also perceive the combined effect of the modification as the addition of a mass term in the denominator of the propagator factor of the previous splitting. In the strongly ordered region, this correction is negligible, whereas in the unordered region, there is a large suppression from the necessity of the propagator in the previous topology having to be very off-shell, which is reflected by the P_{imp} factor. Using the expansion for the unordered region, eq. (D.7), we also see that the effective $2 \to 4$ radiation function, obtained from iterated $2 \to 3$ splittings, is modified as follows,

$$P_{2\to4} \;\propto\; \frac{1}{\hat{Q}^2}\frac{\hat{Q}^2}{Q^2}\frac{1}{Q^2} \;\to\; \frac{1}{Q^4}+\mathcal{O}(\ldots) , \tag{D.9}$$

in the unordered region. That is, the intermediate low scale $\hat{Q}$, is *removed* from the effective $2 \to 4$ function, by the application of the P_{imp} factor.

D.3 All-Orders P_{imp} Factor

The path through phase space taken by an unordered shower history is illustrated in fig. 4, from [4]. An antenna starts showering at a scale equal to its invariant mass, $\sqrt{s}$, and a first $2 \to 3$ branching occurs at the evolution scale $\hat{Q}$. This produces the overall Sudakov factor $\Delta_{2\to3}(\sqrt{s},\hat{Q})$. A daughter antenna, produced by the branching, then starts showering at a scale equal to its own invariant mass, labeled $\sqrt{s_1}$. However, for all scales larger than $\hat{Q}$, the P_{imp} factor suppresses the evolution in this new dipole so that no leading logs are generated. To leading

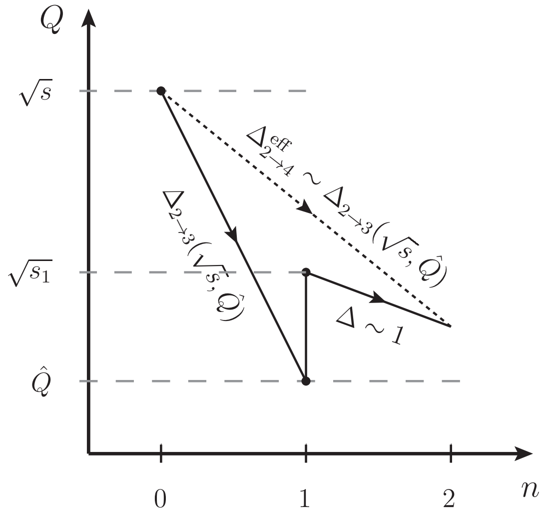

Figure 4: Illustration of scales and Sudakov factors involved in an unordered sequence of two $2 \to 3$ branchings, representing the smoothly ordered shower's approximation to a hard $2 \to 4$ process.

approximation, the effective Sudakov factor for the combined $2 \to 4$ splitting is therefore given by

$$\Delta^{\rm eff}_{2\to4} \;\sim\; \Delta_{2\to3}(\sqrt{s},\hat{Q})\;, \tag{D.10}$$

in the unordered region. Thus, we see that a dependence on the intermediate scale $\hat{Q}$ still remains in the effective Sudakov factor generated by the smooth-ordering procedure. Since $\hat{Q} < Q$ in the unordered region, the effective Sudakov suppression of these points might be "too strong". The smooth ordering therefore allows for phase space occupation in regions corresponding to dead zones in a strongly ordered shower, but it does suggest that a correction to the Sudakov factor may be desirable, in the unordered region.

A study of $Z \to 4$ jets at one loop would be required to shed further light on this question. In the meantime, for all unordered branchings that follow upon a gluon emission, we allow to include a correction to the $\mathrm{P}_{\rm imp}$ factor that removes the leading (eikonal) part of the "extra" Sudakov suppression. We define an all-orders corrected $P_{\rm imp}$ factor as follows:

$$P^{\rm emit}_{\rm imp}\left(Q^2,\hat{Q}^2\right) \to \frac{\alpha_s(Q^2)}{\alpha_s(\hat{Q}^2)}\,\frac{P_{\rm imp}(Q^2,\hat{Q}^2)}{\Delta^{\rm eik}_{2\to3}\left(Q^2,\hat{Q}^2\right)}\;, \tag{D.11}$$

with the Eikonal terms of the Sudakov integral given by [4]:

$$\frac{1}{\Delta^{\rm eik}_{2\to3}\left(Q^2,\hat{Q}^2\right)} \;=\; \exp\left(\frac{\alpha_s(Q^2)}{2\pi}\mathcal{C}\left[I_1(\hat{y})-2I_2(\hat{y})-I_1(y)+2I_2(y))\right]\right)\;,$$

where $\mathcal{C}$ is the colour factor of the first $2\to3$ branching (the one that produced the intermediate scale $\hat{Q}$) inside which the unordered $2\to4$ branching is occurring, and $y = Q^2/m_2^2$ ($\hat{y} = \hat{Q}^2/m_2^2$)

is the branching scale normalized to the invariant mass squared of that antenna. For evolution in $p_\perp$ (the default for gluon emissions), the I_1 and I_2 integrals are [4]:

$$I_1(y) = \frac{\pi^2}{6} + \frac{1}{2}\ln^2\left[\frac{y^2}{2\left(1+\sqrt{1-y^2}\right)-y^2}\right] - \ln^2\left[\frac{1}{2}\left(1+\sqrt{1-y^2}\right)\right] - 2\,\mathrm{Li}_2\left[\frac{1}{2}\left(1+\sqrt{1-y^2}\right)\right] \tag{D.12}$$

$$I_2(y) = -\left(\ln\left[\frac{y^2}{2\left(1+\sqrt{1-y^2}\right)-y^2}\right] + 2\sqrt{1-y^2}\right) \;, \tag{D.13}$$

with expressions for other ordering types available in [4]. We note that this factor neglects (positive) collinear-singular terms and (positive) corrections from the running of α_s between Q and $\hat{Q}$, hence we expect that even this correction factor still only represents a partial compensation, at a level equivalent to removing spurious terms of total order $\alpha_s^3\ln^3(\hat{Q}^2/Q^2)$ and $\alpha_s^3\ln^2(\hat{Q}^2/Q^2)$. We also note that a similar factor could be applied

D.4 Gluon Splitting: The Ariadne Factor

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D.5 Matrix-Element Corrections: Leading Colour

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D.6 Matrix-Element Corrections: Full Colour

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D.7 Matrix-Element Corrections: MHV amplitudes

For fast evaluation of certain types of helicity configuration VINCIA uses Maximally Helicity Violating (MHV) amplitudes. MHV amplitudes have the advantage of being an analytical solution for n partons which is independent of Feynman diagrams. In the following we consider all particles to be outgoing and massless.

To be in the MHV configuration all but two particles must have the same helicity. We define our spinors as:

$$u_\pm(p) = \frac{1}{2}\left(1\pm\gamma^5\right)u(p)\;, \qquad \overline{u_\pm(p)} = \overline{u(p)}\frac{1}{2}\left(1\mp\gamma^5\right)\;, \tag{D.14}$$

together with their inner products:

$$\overline{u_-(i)}u_+(j) \equiv \langle ij \rangle = \sqrt{p_j^+}e^{i\phi_i} - \sqrt{p_i^+}e^{i\phi_j} \;, \tag{D.15}$$

$$\overline{u_+(i)}u_-(j) \equiv [ij] = \langle ji \rangle^* \;, \tag{D.16}$$

where $p_i^+ = p_i^0 + p_i^3$ and $e^{i\phi_i} = \frac{p_i^1 + ip_i^2}{\sqrt{p_i^+}}$. For more details about spinor inner products and their properties see [5]. Note that in recent literature one often finds the convention $[ij] = \langle ij \rangle^*$, which is different to above. Any future formulae/spinor-helicity properties borrowed from literature should bare this in mind.

The MHV amplitudes used in VINCIA are all colour-ordered. We use a different QCD convention for MHV than in the rest of VINCIA. For MHV amplitudes the QCD Casimirs become $T_R = 1$, $C_F = 8/3$, and $C_A = 3$, which affects the colour-algebra as seen in [5]. The n-gluon MHV amplitude with negative-helicity gluons at positions i and j is given by the Parke Taylor formula [6]:

$$A_n(i^-, j^-) = i\frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} \;. \tag{D.17}$$

There is an equally compact MHV formula for the process involving $n-2$ gluons and a quark anti-quark pair. If the quark and gluon i each have negative helicity, and the anti-quark and all other gluons have positive helicity, then the amplitude is [7]:

$$A_n(q^-, i^-, \bar{q}^+) = \frac{\langle qi \rangle^3 \langle \bar{q}i \rangle}{\langle \bar{q}q \rangle \langle q1 \rangle \langle 12 \rangle \ldots \langle (n-2)\bar{q} \rangle} \;, \tag{D.18}$$

where the numbers refer to the (colour ordered) gluons. If we exchange the helicities on the quarks, it is sufficient to exchange the exponents in the numerator. See [7] for the four-quark, n-gluon MHV amplitude; as well as the two-quark, two-lepton, n-gluon MHV amplitude. For each amplitude, exchanging the helicity of each particle corresponds to exchaning $\langle ij \rangle \to [ji]$.

The MHV amplitudes involving the exchange of a W boson still need testing, but have been written into MHV.cc

When doing the helilticy-clustering, an MHV configuration will always cluster back into either an unphysical helicity state, or into an MHV state. This allows for quick matrix-element corrections of complex states such as the 7-gluon state. The MHV configurations also provide the dominant contributions to a helicity-summed amplitude. It may therefore be useful to give the user the option to include MHV corrections for very high-multiplicity states.

D.8 Matrix-Element Corrections: Different Interfering Borns

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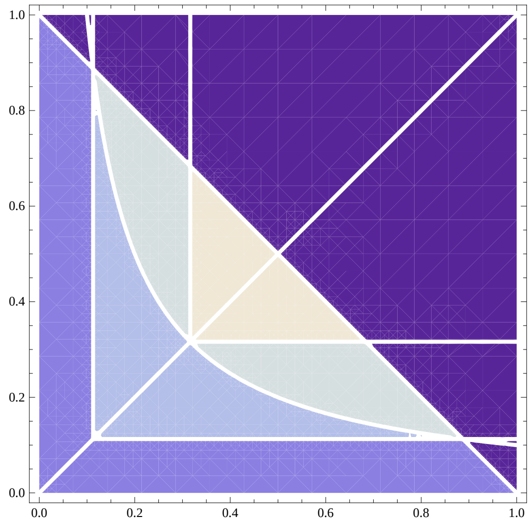

Figure 5: Illustration of the intersection/nesting of p_T and m_D contours.

E Cutoff Boundaries

E.1 Fixed Transverse Momentum

- Consider the region defined by $y_{ij}y_{jk} \geq y_\perp$. For illustration, a value of $y_\perp = 0.1$ was used for the contour shown with light green shading in fig. 5.
- The (larger) invariant-mass region that completely encloses the $y_\perp$ one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \frac{1}{2}(1 - \sqrt{1 - 4y_\perp})$. This is shown with light blue shading in fig. 5.
- The (smaller) invariant-mass region that is completely enclosed by the $y_\perp$ one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \sqrt{y_\perp}$. This is shown with light yellow shading in fig. 5.

To translate this to evolution variables, with arbitrary normalization factors, use $y_\perp = Q_\perp^2/s_{IK}/N_\perp$ and $m_D^2/s_{IK}/N_D$.

E.2 Fixed Dipole Mass

- Consider the region defined by $\min(y_{ij}y_{jk}) \geq y_D$, with y_D some fixed value.
- The (larger) transverse-momentum region that completely encloses the y_D one is defined by $y_\perp = y_{ij}y_{jk} \geq y_D^2$. This relationship is illustrated by the light-green and light-yellow shaded regions in fig. 5.
- The (smaller) transverse-momentum region that is completely enclosed by the y_D one is defined by $y_\perp = y_{ij}y_{jk} \geq \frac{1}{4}(1 - (1 - 2y_D)^2)$. This relationship is illustrated by the light-green and light-blue shaded regions in fig. 5.

To translate this to evolution variables, with arbitrary normalization factors, use $y_\perp = Q_\perp^2/s_{IK}/N_\perp$ and $m_D^2/s_{IK}/N_D$.

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