Does an Even Split in One Suit Mean that Other Suits are More Likely to Split Evenly?

Hearts Card Game: An R Language Program to Track Bayesian Probabilities Based Upon Inferences During the Play of the Hand

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Summary

We quantitatively evaluated the assertion that says if one suit is found to be evenly distributed among the 4 players, the rest of the suits are more likely to be evenly distributed. Our mathematical analyses show that, if one suit is found to be evenly distributed, then a second suit has a slightly elevated probability (ranging between 10% to 15%) of being evenly distributed. If two suits are found to be evenly distributed, then a third suit has a substantially elevated probability (ranging between 30% to 50%) of being evenly distributed. The probabilities for more unbalanced distributions decrease in similar proportion, but these probabilities are quite low to start with, so the further decreases may only rarely be of practical significance.

Motivation

There is a bit of conventional wisdom in bridge and hearts, that says if one suit is found to be evenly distributed among the 4 players, the rest of the hand is more likely to be evenly distributed. This was remarked upon by Ely Culbertson [1,2]:

> The Law of Symmetry can be defined as a guide for judging the types (balanced and unbalanced) of suit and hand-patterns in the remaining three hands or, if two hands are seen, in the two unknown hands.
> . . .
>
> When, however, you hold a hand-pattern belonging to the respectable but rather bourgeois family of balanced patterns, say the prosaic 5-3-3-2, this pattern symptom is nothing that is really alarming. At least one of the unknown hands, and at least one of the suits, will also be balanced, and probably other hands and suits as well.

Here we are going to use a mathematical analysis to quantify this assertion.

Methods

The methods used here, embodied in my custom R functions, represent an extension of the studies that I had reported in the section entitled "Probability of a Player Holding a Given Number of Cards of a Given Suit (*i.e.,* Spades)" in the document "Drawing Inferences." [3,4].

I applied a standard method in mathematical statistics to determine the expected distributions of suits, among the hands of 4 players, for random deals of a deck of cards. This distribution is conceptually different than the distribution the 4 suits within a single player's 13 cards. However, surprisingly, the same result holds for both of these distributions [5-7].

My computations are not novel, in fact they are well known. But I wanted to make sure that my custom R functions were accurate, by comparing my results with the authentic values given in Table 3 of reference [3]. Then I would have confidence in the novel values that I compute when carrying out the studies mentioned in the summary section above.

There are two types of situations that I study here. The first situation involves a standard 52 card deck, without any *a priori* information about suit distribution. The second situation involves a fictitious 40 card deck, that arises from a 52 card deck after 3 tricks have been played, so that 12 of the 52 cards are no longer in play. In this scenario, all 12 cards are of the same suit (say spades, for instance), so the remaining 40 card deck is comprised of 13 cards each of the other suits, plus one left-over spade. This same logic was extended to a 28 card deck and finally (and trivially) to a 16 card deck.

The formula for the standard 52 card deck scenario was derived using the principles in reference [4] and is exemplified here for a standard 52 card deck to compute the probability of a 3-3-3-4 suit distribution:

```
NUMERATOR<- choose(13,3)*choose(39,10)*choose(10,3)*choose(29,10)*choose(7,3)*choose(19,10)

DENOMINATOR<-choose(52,13)*choose(39,13)*choose(26,13)

PERMUTATIONS<-4

PROBABILTY<-PERMUTATIONS * NUMERATOR/DENOMINATOR
```

PERMUTATIONS was set to 4, since the 3-3-3-4 distribution can be achieved in 4 different ways depending on which of the 4 players hold the 4 card suit.

The organization of the NUMERATOR and DENOMINATOR are detailed in Tables 1 and 2, respectively.

The 40 card deck is equivalent to a standard 52 card deck after 3 tricks for which all of the 12 cards played were of the same suit, say spades. This leaves a 40 card deck of 3 complete suits and 1 left-over spade. So the effect of observing that one suit is evenly distributed can be analyzed by applying the same formalism to a 40 card deck as we just demonstrated for a 52 card deck.

```
NUMERATOR <-choose(13,3)*choose(27,7) * choose(10,3)*choose(20,7) * choose(7,3)*choose(13,7)

DENOMINATOR <-choose(40,10)*choose(30,10)*choose(20,10)

PERMUTATIONS<-4

PROBABILTY<-PERMUTATIONS * NUMERATOR/DENOMINATOR
```

The organization of the NUMERATOR and DENOMINATOR are detailed in Tables 3 and 4, respectively.

The formalism can be generalized for any acceptable deck size and any desired distribution of 13 cards, as described in Tables 5 – 7. These expressions can be used to construct an R function to compute the numerator and denominator. The permutations can be computed by using the R function *permn()* in the package *combinat*.

RESULTS AND DISCUSSION

The validation of my in-house R functions can be obtained by comparing the numerical values of the probability for deck size 52 (Table 8) with the corresponding authentic values tabulated in Table 3 of reference 3.

The overall results are tabulated in Figure 1 and shown graphically Table 8. To make it easier to cross-reference the figure and the table, I have annotated the figure with the 5 most balanced (and most significant) distributions.

The unique numerical value representing the degree of unbalanced suit distribution was given by the standard deviation of the 4 numerical values designating the suit distribution across the four players' hands. The distribution with the lowest standard deviation (0.50) was 4-3-3-3 (see the column labelled "Balance" in Table 8).

The main result can be gleaned by comparing the probability for the 4-3-3-3 distribution between the 52 card deck and the 40 card deck. Recall that the 40 card deck is just the 52 card deck after the observation that the first suit, say spades, is evenly divided in the first 3 tricks. We can see that the probability for 4-3-3-3 increases from the "standard" value of 0.105 to 0.120, corresponding to a 15% increase, as tabulated in the column "Fractional Change for Deck Size 40." Not surprisingly, the increase is much higher (52%) after 2 suits had been seen to be divided evenly.

Relationship to the Main Paper on Inference

A secondary reason for undertaking this study is to determine if the observation of a suit breaking evenly might be used for drawing inferences about the distribution of the remaining cards. In particular, should any special action be taken in adjusting the values in the probs matrix, beyond the automatic or manual adjustments?

If the effect if small, then there is little to be gained by explicitly performing a special adjustment. But the effect of a 15% change might be significant enough to consider.

The other aspect to consider is whether the existing method of adjusting the probs matrix implicitly includes this adjustment, “without realizing it?”

Finally, even if an adjustment were called for, it is not clear as to what adjustment could be made. After all, the adjustments are made one at a time to a particular card, not to an entire distribution.

RELEVANT IN-HOUSE CUSTOM R PROGRAMS

R Program	Function
compareProbs()	organizes invoking evenBreakDriver() for 4 deck sizes, and presenting the results in a table and in a graph
evenBreakDriver()	loop through all possible distributions of a single suit across 4 hands
evenBreak()	compute the probability of a given distribution of a single suit across 4 hands

These programs are located in the file *evenBreak.R*

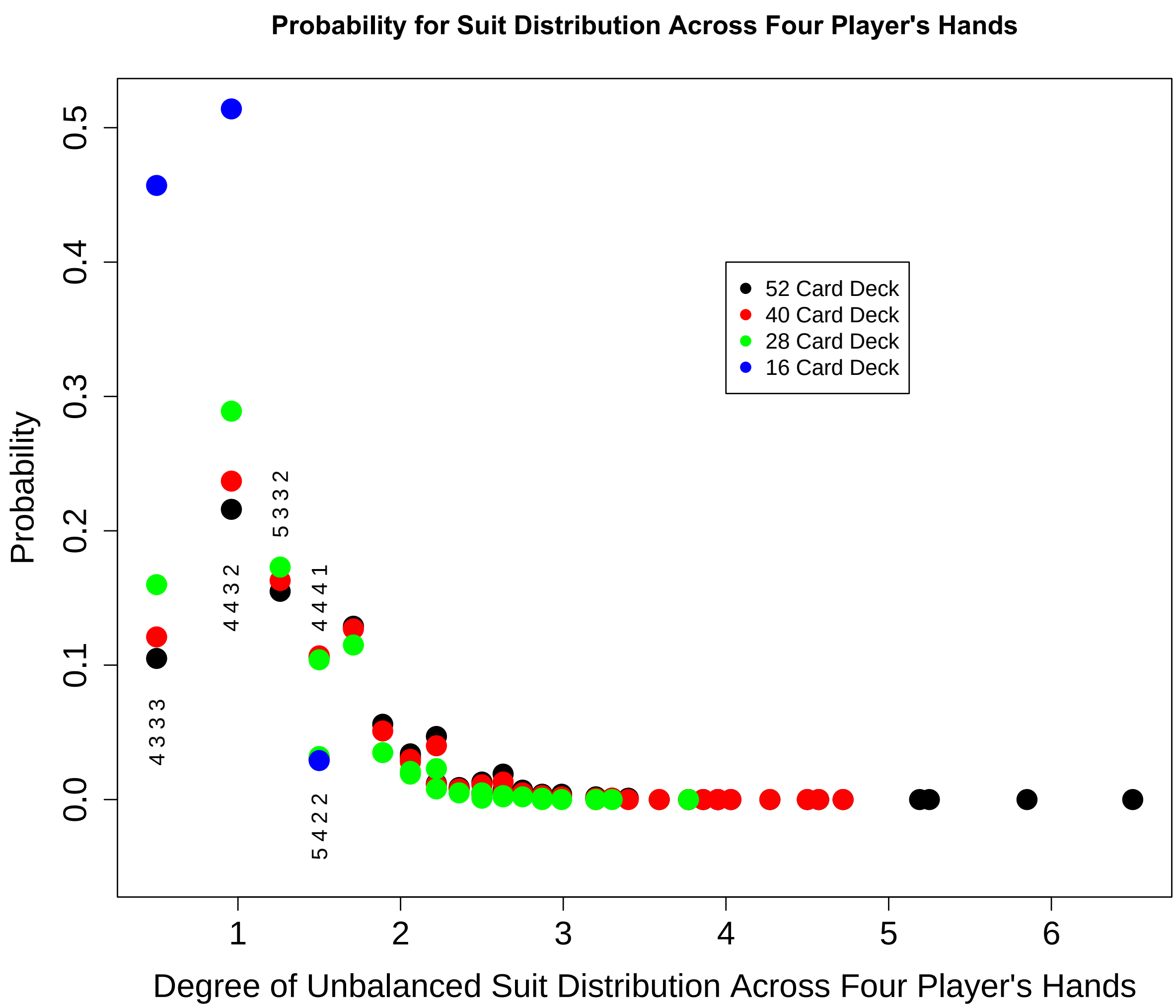

Figure 1. The extent to which an even split in one suit mean that other suits are more likely to split evenly.

Footnotes applying to all/multiple tables:

1. Without loss of generality and for the sake of clarity, arbitrarily use spades as the example of the suit whose distribution is being computed.

2. The numbers of cards are color-coded to make it easier to identify the same number used in different contexts.

3. For Tables 3 – 7, The 40 card deck arises from the fact that we are assuming *a priori* that the distribution of a suit other than spades, say diamonds, was seen to break 3-3-3-3 during the play of the cards. In effect, the remainder of the game is being played with 39 cards comprising clubs, hearts, and spades, and a single diamond.

4. Player 4 was omitted, as there was no "choice" remaining, player 4 simply is required to take whatever cards are left over. This would result in adding an expression that amounts to a factor of 1.

Table 1. Organization of the Expression for the Numerator for a 52 Card Deck and a 3-3-3-4 Suit Distribution Across the 4 Players' Hands

Entry	**Term**	**Player**	**Explanation**
1	choose(13, 3)	1	There are 13 spades in the deck. Choose any 3 spades.
2	choose(39, 10)		There are 52 – 13 = 39 non-spades in the deck. 3 spades have been chosen for this hand. Choose any 10 of the 39 non-spades to complete the 13 card hand.
3	choose(10, 3)	2	There are 13 – 3 = 10 spades in the deck that have not been chosen yet. Choose any 3 of these.
4	choose(29, 10)		There are 39 – 10 = 29 non-spades in the deck that have not been chosen yet. 3 spades have been chosen for this hand. Choose any 10 of the 29 non-spades to complete the 13 card hand.
5	choose(7, 3)	3	There are 10 – 3 = 7 spades in the deck that have not been chosen yet. Choose any 3 of these.
6	choose(19,10)		There are 29 – 10 = 19 non-spades in the deck that have not been chosen yet. 3 spades have been chosen for this hand. Choose any 10 of the 19 non-spades to complete the 13 card hand.

Table 2. Organization of the Expression for the Denominator for a 52 Card Deck and a 3-3-3-4 Suit Distribution Across the 4 Players' Hands

Entry	**Term**	**Player**	**Explanation**
1	choose(52,13)	1	There are 52 cards in the deck. Pick any 13 to comprise the hand for player 1.
2	choose(39,13)	2	There are 52 - 13 = 39 cards in the deck. Pick any 13 to comprise the hand for player 2.
3	choose(26,13)	3	There are 39 - 13 = 26 cards in the deck. Pick any 13 to comprise the hand for player 1.

Table 3. Organization of the Expression for the Numerator for a 40 Card Deck and a 3-3-3-4 Suit Distribution Across the 4 Players' Hands

Entry	Term	Player	Explanation[1]
1	choose(13, 3)	1	There are 13 spades in the deck. Choose any 3 spades.
2	choose(27, 7)		There are 40 – 13 = 27 non-spades in the deck. 3 spades have been chosen for this hand. Choose any 7 of the 27 non-spades to complete the 10 card hand.
3	choose(10, 3)	2	There are 13 – 3 = 10 spades in the deck that have not been chosen yet. Choose any 3 of these.
4	choose(20, 7)		There are 27 – 7 = 20 non-spades in the deck that have not been chosen yet. 3 spades have been chosen for this hand. Choose any 7 of the 20 non-spades to complete the 10 card hand.
5	choose(7, 3)	3	There are 10 – 3 = 7 spades in the deck that have not been chosen yet. Choose any 3 of these.
6	choose(13,7)		There are 20 – 7 = 13 non-spades in the deck that have not been chosen yet. 3 spades have been chosen for this hand. Choose any 7 of the 13 non-spades to complete the 10 card hand.

Table 4. Organization of the Expression for the Denominator for a 40 Card Deck and a 3-3-3-4 Suit Distribution Across the 4 Players' Hands

Entry	**Term**	**Player**	**Explanation**
1	choose(40,10)	1	There are 40 cards in the deck. Pick any 10 to comprise the hand for player 1.
2	choose(30,10)	2	There are 40 - 10 = 30 cards in the deck. Pick any 10 to comprise the hand for player 2.
3	choose(20,10)	3	There are 30 - 10 = 20 cards in the deck. Pick any 10 to comprise the hand for player 1.

Table 5. Definitions of Symbolic Expressions for a D Card Deck and an n1-n2-n3-n4 Suit Distribution Across the 4 Players' Hands

Symbol #	**Symbol**	**Definition**	**Examples**	
1	D	Deck size	52	40
2	D4	One quarter of deck size	13	10
3	n1	Number of spades in player 1 hand	3	3
4	n2	Number of spades in player 2 hand	3	3
5	n3	Number of spades in player 3 hand	3	3
6	n4	Number of spades in player 4 hand	4	4

Table 6. Organization of the Expression for the Numerator for a D Card Deck and an n1-n2-n3-n4 Suit Distribution Across the 4 Players' Hands

Entry	Player	Term	52 Card Deck and a 3-3-3-4 Suit Distribution		40 Card Deck and a 3-3-3-4 Suit Distribution	
			Plug into formula	**Confirmation from Table 1**	**Plug into formula**	**Confirmation from Table 3**
1	1	choose(13,n1)	choose(13,3)	choose(13, 3)	choose(13,3)	choose(13, 3)
2		choose(D – 0*D4 – 13, D4 – n1)	choose(52 – 13 ,13 - 3)	choose(39, 10)	choose(40 – 13 ,10 - 3)	choose(27, 7)
3	2	choose(13 – n1, n2)	choose(13 - 3,3)	choose(10, 3)	choose(13 - 3,3)	choose(10, 3)
4		choose(D – 1*D4 – (13 – n1), D4 – n2)	choose(52 – 13 – (13 – 3),13 - 3)	choose(29, 10)	choose(40 – 10 – (13 – 3),10 - 3)	choose(20, 7)
5	3	choose(13 – n1 – n2, n3)	choose(13 - 3 - 3,3)	choose(7, 3)	choose(13 - 3 - 3,3)	choose(7, 3)
6		choose(D – 2*D4 – (13 – n1 – n2), D4 – n3)	choose(52 – 26 – (13 – 3 – 3),13 - 3)	choose(19,10)	choose(40 – 20 – (13 – 3 – 3),10 - 3)	choose(13,7)

Table 7. Organization of the Expression for the Denominator for a D Card Deck and an n1-n2-n3-n4 Suit Distribution Across the 4 Players' Hands

Entry	**Player**	**Term**	**52 Card Deck and a 3-3-3-4 Suit Distribution**	**40 Card Deck and a 3-3-3-4 Suit Distribution**
1	1	choose(D – 0*D4,D4)	choose(52 – 0*13,13)	choose(40 – 0*10,10)
2	2	choose(D – 1*D4,D4)	choose(52 – 1*13,13)	choose(40 – 1*10,10)
3	3	choose(D – 2*D4,D4)	choose(52 – 2*13,13)	choose(40 – 2*10,10)

Table 8. Probabilities for the distribution of a given suit across four hands, conditioned on the number of suits that are known to be evenly distributed.

n1	n2	n3	n4	Balance	Probability for Deck Size				Fractional Change for Deck Size	
					52	40	28	16	40	28
4	3	3	3	0.5	0.105	0.121	0.16	0.457	0.15	0.52
4	4	3	2	0.96	0.216	0.237	0.289	0.514	0.10	0.34
4	4	4	1	1.5	0.03	0.031	0.032	0.029	0.03	0.07
5	3	3	2	1.26	0.155	0.163	0.173	-	0.05	0.12
5	4	2	2	1.5	0.106	0.107	0.104	-	0.01	-0.02
5	4	3	1	1.71	0.129	0.127	0.115	-	-0.02	-0.11
5	4	4	0	2.22	0.012	0.011	0.008	-	-0.08	-0.33
5	5	2	1	2.06	0.032	0.028	0.021	-	-0.13	-0.34
5	5	3	0	2.36	0.009	0.008	0.005	-	-0.11	-0.44
6	3	2	2	1.89	0.056	0.051	0.035	-	-0.09	-0.38
6	3	3	1	2.06	0.034	0.03	0.019	-	-0.12	-0.44
6	4	2	1	2.22	0.047	0.04	0.023	-	-0.15	-0.51
6	4	3	0	2.5	0.013	0.011	0.005	-	-0.15	-0.62
6	5	1	1	2.63	0.007	0.005	0.002	-	-0.29	-0.71
6	5	2	0	2.75	0.007	0.005	0.002	-	-0.29	-0.71
6	6	1	0	3.2	0.001	0	0	-	-1.00	-1.00
7	2	2	2	2.5	0.005	0.004	0.001	-	-0.20	-0.80
7	3	2	1	2.63	0.019	0.013	0.003	-	-0.32	-0.84
7	3	3	0	2.87	0.003	0.002	0	-	-0.33	-1.00
7	4	1	1	2.87	0.004	0.003	0.001	-	-0.25	-0.75
7	4	2	0	2.99	0.004	0.002	0	-	-0.50	-1.00

Deck sizes 52, 40, 28, and 16 refer to 0, 1, 2, or 3 suits known to be evenly distributed.

REFERENCES

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7. Ken Monzingo, Hand and Suit Patterns, http://web2.acbl.org/documentlibrary/teachers/celebritylessons/handpatternsrevised.pdf.